Furusho's on A Betti Counterpart of the Harmonic Coproduct, I

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A Betti counterpart of the harmonic coproduct, I
joint w/ Benjamin Enriquez.

1. Harmonic co-product in the de Rham setting.
2. — " — Betti setting

MZV: $m, k_1 \dots k_m \in \mathbb{N}$

$$\zeta(k_1,\dots,k_m) := \sum_{0<n_1<\dots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

conv. if $k_m>1$.

<u>Prop</u> MZV are closed under multiplication.

<u>Pf1</u> (Harmonic product) (stuffle)

e.g. $\zeta(a)\zeta(b) = \zeta(a,b)+\zeta(a+b)+\zeta(b,a)$

<u>Pf2</u> (Shuffle product)

$$\zeta(a)\zeta(b) = \left(\int_{0<s_1<\dots<s_a<1} \frac{ds_1}{1-s_1}\wedge\frac{ds_2}{s_2}\wedge\dots\wedge\frac{ds_a}{s_a}\right)\left(\text{same w/ } b.\right)$$

by interlacing the variables:

$$= \sum_{i=0}^{a-1}\binom{b-1+i}{i}\zeta(a-i,b+i) + \sum_{j=0}^{b-1}\binom{a-1+j}{j}\zeta(b-j,a+j)$$

The "double shuffle relation" is the equality of the RHS's of the two proofs.

Racinet's formulation (02)

$$\mathcal{V}_{DR} = \mathbb{C}\langle x_0, x_1 \rangle$$ [the free associative algebra]

$$\deg x_i = 1$$

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$$

$$\mathcal{W}_{DR} = \mathcal{V}_{DR} \big/ \mathcal{V}_{DR} \cdot x_0 \xleftarrow{\pi_Y} \mathcal{V}_{DR}$$

$$\cong$$

$$\mathbb{C} \oplus \mathcal{V}_{DR} \cdot x_1$$

$$\cong$$

$$\mathbb{C}\langle y_1, y_2, y_3 \dots \rangle \quad \text{w/ } y_n = x_0^{n-1} x_1$$

$$\deg y_n = n \qquad \Delta_*(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k+l=n} y_k \otimes y_l$$

The harmonic co-product.

Can complete everything...

<u>Notation</u> For $\varphi \in \hat{\mathcal{V}}_{DR}$

coeff of $x_0^{n-1} x_1$ in φ

$$\Gamma_\varphi := \exp\left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\varphi | x_0^{n-1} x_1) x_1^n \right\}$$

$$\Theta(\varphi) = \Gamma_\varphi \varphi \in \hat{\mathcal{V}}_{DR}$$

$$\varphi_* = \pi_Y(\Theta(\varphi)) \in \hat{\mathcal{W}}_{DR}$$

Corresponds to shuffle product

Def For $\mu \in \mathbb{C}$ set

$$DMR_\mu := \left\{ \varphi \in \hat{\mathcal{U}}^\times_{DR} : \begin{array}{l} \Delta(\varphi) = \varphi \otimes \varphi \\ \Delta_*(\varphi) = \varphi_* \otimes \varphi_* \\ (\varphi|x_0) = (\varphi|x_1) = 0 \\ (\varphi|x_0x_1) = -\mu^2/24 \end{array} \right\}$$

double Melangé et regularization

corresponds to harmonic product.

Thm (Racinet)

① $\Phi_{KZ} \in DMR_\mu$ w/ $\mu = 2\pi i$ and

$$\Phi_{KZ} = 1 + \sum_m \sum_{\substack{k_1 \dots k_m \in \mathbb{N} \\ k_m > 1}} \zeta(k_1 \dots k_m)\, x_0^{k_m - 1} x_1 \dots x_0^{k_1 - 1} x_1 + \begin{pmatrix}\text{regularized} \\ \text{terms}\end{pmatrix} \in \hat{\mathcal{U}}^*_{DR}$$

② DMR_0 forms a group under

$$\varphi_1 \circledast \varphi_2 = S_{\varphi_1}(\varphi_2) := \\ = \varphi_1(x_0, x_1)\, \varphi_2(x_0, \varphi_1^{-1} x_1 \varphi_1)$$

③ For $\mu \neq 0$, DMR_μ forms a left DMR_0-torsor under the above multiplication. For $\varphi_1 \in DMR_0$ $\varphi_2 \in DMR_\mu$

Remark:

$$GRT_1 \circlearrowright M_1 \circlearrowleft GT_1 \qquad \text{Drinfel'd}$$

equality conjectural $\int$ by Furusho $\overset{?}{=}$

$$DMR_0 \circlearrowright DMR_1 \circlearrowleft \square \qquad \text{us}$$

$DMR_0 \circlearrowright DMR_1 \circlearrowleft \mho$ us

Today.

How 2

$TM = (Gp\hat{\mathcal{V}}_{DR}, \circledast) > DMR_0$

explicit

$TM \longrightarrow Aut_{\mathbb{Q}\text{-vect}} \hat{\mathcal{V}}_{DR}$

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