Pensieve header: The 1D Zip Algebra.

ZipDemo

In[∘]:=
```
{x*, y*, z*} = {ξ, η, ζ}; {ξ*, η*, ζ*} = {x, y, z};
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] := (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :→ ∂_{ζ*,d} f) /. ζ* → 0
```

ZipDemo

In[∘]:=
```
{Zip_{ζ}[ζ^2 e^(δ z^2)], Zip_{ζ}[ζ^4 e^(δ z^2)]}
```

ZipDemo

Out[∘]= $\{2\,\delta,\ 12\,\delta^2\}$

GZip

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
𝔼 /: Zip_{ζs_List}@𝔼[Q_, P_] :=      (* 𝔼[Q,P] means e^Q P *)
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule; Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^(-Q2) Zip_{ζs}[e^Q1 (P /. zrule)]] ];
```

The Algebra

In[∘]:=
```
Table[Zip_{η}[𝔼[ξ y + η z, (ξ y)^m (η z)^n]], {m, 0, 3}, {n, 0, 3}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \mathbb{E}[z\,\xi,\,1] & \mathbb{E}[z\,\xi,\,z\,\xi] & \mathbb{E}[z\,\xi,\,z^2\,\xi^2] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3] \\ \mathbb{E}[z\,\xi,\,z\,\xi] & \mathbb{E}[z\,\xi,\,z\,\xi\,(1+z\,\xi)] & \mathbb{E}[z\,\xi,\,z^2\,\xi^2\,(2+z\,\xi)] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3\,(3+z\,\xi)] \\ \mathbb{E}[z\,\xi,\,z^2\,\xi^2] & \mathbb{E}[z\,\xi,\,z^2\,\xi^2\,(2+z\,\xi)] & \mathbb{E}[z\,\xi,\,z^2\,\xi^2\,(2+4\,z\,\xi+z^2\,\xi^2)] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3\,(6+6\,z\,\xi+z^2 \\ \mathbb{E}[z\,\xi,\,z^3\,\xi^3] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3\,(3+z\,\xi)] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3\,(6+6\,z\,\xi+z^2\,\xi^2)] & \mathbb{E}[z\,\xi,\,z^3\,\xi^3\,(6+18\,z\,\xi+9\,z^2\,\xi \end{pmatrix}$$

In[∘]:=
```
Table[Last@Zip_{η}[𝔼[ξ y + η z, (ξ y)^m (η z)^n]] /. ξ → 1, {m, 0, 4}, {n, 0, 4}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ z & z\,(1+z) & z^2\,(2+z) & z^3\,(3+z) & z^4\,(4+z) \\ z^2 & z^2\,(2+z) & z^2\,(2+4\,z+z^2) & z^3\,(6+6\,z+z^2) & z^4\,(12+8\,z+z^2) \\ z^3 & z^3\,(3+z) & z^3\,(6+6\,z+z^2) & z^3\,(6+18\,z+9\,z^2+z^3) & z^4\,(24+36\,z+12\,z^2+z^3) \\ z^4 & z^4\,(4+z) & z^4\,(12+8\,z+z^2) & z^4\,(24+36\,z+12\,z^2+z^3) & z^4\,(24+96\,z+72\,z^2+16\,z^3+z^4) \end{pmatrix}$$

In[∘]:=
```
B[n_ over k_] := Binomial[n, k];
f[m_, n_] := Simplify[Sum_{k=0}^{Min[m,n]} k! B[m over k] B[n over k] z^(m+n-k)];
Table[f[m, n], {m, 0, 4}, {n, 0, 4}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ z & z(1+z) & z^2(2+z) & z^3(3+z) & z^4(4+z) \\ z^2 & z^2(2+z) & z^2(2+4z+z^2) & z^3(6+6z+z^2) & z^4(12+8z+z^2) \\ z^3 & z^3(3+z) & z^3(6+6z+z^2) & z^3(6+18z+9z^2+z^3) & z^4(24+36z+12z^2+z^3) \\ z^4 & z^4(4+z) & z^4(12+8z+z^2) & z^4(24+36z+12z^2+z^3) & z^4(24+96z+72z^2+16z^3+z^4) \end{pmatrix}$$

A Module

In[∘]:=
```
Table[Zip_{η}[𝔼[ξ y, (ξ y)^m (η z)^n]], {m, 0, 3}, {n, 0, 3}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \mathbb{E}[0,1] & \mathbb{E}[0,z\xi] & \mathbb{E}[0,z^2\xi^2] & \mathbb{E}[0,z^3\xi^3] \\ \mathbb{E}[0,0] & \mathbb{E}[0,z\xi] & \mathbb{E}[0,2z^2\xi^2] & \mathbb{E}[0,3z^3\xi^3] \\ \mathbb{E}[0,0] & \mathbb{E}[0,0] & \mathbb{E}[0,2z^2\xi^2] & \mathbb{E}[0,6z^3\xi^3] \\ \mathbb{E}[0,0] & \mathbb{E}[0,0] & \mathbb{E}[0,0] & \mathbb{E}[0,6z^3\xi^3] \end{pmatrix}$$

In[∘]:=
```
Table[Last@Zip_{η}[𝔼[ξ y, (ξ y)^m (η z)^n]] /. ξ → 1, {m, 0, 4}, {n, 0, 4}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & z & z^2 & z^3 & z^4 \\ 0 & z & 2z^2 & 3z^3 & 4z^4 \\ 0 & 0 & 2z^2 & 6z^3 & 12z^4 \\ 0 & 0 & 0 & 6z^3 & 24z^4 \\ 0 & 0 & 0 & 0 & 24z^4 \end{pmatrix}$$

In[∘]:=
```
Expand[y (y - 1) * y (y - 1) (y - 2) ==
  y (y - 1) (y - 2) (y - 3) (y - 4) + 6 y (y - 1) (y - 2) (y - 3) + 6 y (y - 1) (y - 2)]
```

Out[∘]= True