MAT 247 for Kudla: Inner products, Orthogonal Complements, etc.

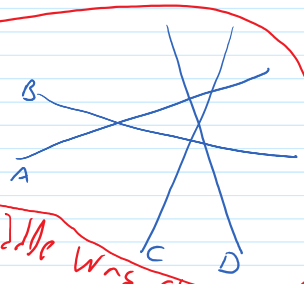
January 31, 2018 11:26 AM

@SF 1105

Kudla: "I finished proving Theorem 6.5 but did not have time to do the Corollary on p.347 or the Definition on p 349. So you could start with these."

DROR BAR-NATAN for Steve Kudla, one time only.

Maybe: 4 cars drive on straight lines & constant speed in the Sahara desert. A meets B, C, & D, B meets A, C, D. Will C meet D?



Riddle was skipped.

Inner product: $\langle -,- \rangle : V \times V \to F \,(=\mathbb{R} \text{ or } \mathbb{C})$ s.t.

1. Sesquilinear: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, y \rangle + \beta \langle y, z \rangle$

$$\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$$

2. Symmetric/Hermitian: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

3. positive-definite:

$\langle x, x \rangle \geq 0$, equality iff $x = 0$.

motivation from $\mathbb{R}^3$, introduce $\|\cdot\|$

I thought this would be a quick comment, but I ended up taking longer.

<u>Examples:</u> $\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right\rangle = \sum a_i \bar{b}_i$ $\quad \langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \bar{g}$ $\quad \langle A, B \rangle = \operatorname{tr} B^* A$

"conjugate transpose"

C-S. $|\langle x, y \rangle| \leq \|x\| \|y\|$ $\quad \Delta: \|x+y\| \leq \|x\| + \|y\|$

<u>Def</u> perpendicular $\langle v, w \rangle = 0$ $\quad$ orthogonal subset $i \neq j \; \langle v_i, v_j \rangle = 0$ $\quad$ orthonormal subset $\langle v_i, v_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\Rightarrow$ Pythagoras: $\|v+w\|^2 = \|v\|^2 + \|w\|^2$

much of this was supposed to be "blue", but the class before me ended late so I had to write it later, and this took some extra time.

<u>Theorem</u> (G-S) Given $S = \{w_1 \ldots w_n\}$ a lin-indep subset of $(V, \langle \rangle)$, there exists an orthonormal $S' = \{v_1 \ldots v_n\}$ s.t.

$$\forall k \leq n \quad \operatorname{span}\{w_1 \ldots w_k\} = \operatorname{span}\{v_1 \ldots v_k\}$$

<u>Theorem</u> Every f.d. inner product space has an O.N basis. If $\beta = \{v_1, \ldots v_n\}$ is such & $x \in V$, then

$$x = \sum \underbrace{\langle x, v_i \rangle}_{\alpha_i \text{ "Fourier coefficients"}} v_i$$

I did not prove anything here, just reviewed.

<u>Example</u> Under suitable conditions, $f = \sum a_n e^{inx}$ where $a_n = \frac{1}{2\pi} \int f(x) e^{-inx} dx$

I explained this was an ∞-dim aside which gets justified in analysis classes.

classes. People asked, so I mentioned w/o detail that there are discrete analogs, and also the Fourier transform.

<u>Claim</u> As above, if $x=\sum \alpha_i v_i$ & $y=\sum \beta_i v_i$ then $\langle x,y\rangle = \sum \alpha_i \bar\beta_i$.

[So "there is only one inner product space in each dim"]

<u>Corollary</u> Given $(V,\langle\rangle)$ & O.N $\beta=\{v_1\dots v_n\}$ & linear $T:V\to V$. Let $A=[T]_\beta$. Then

$$A_{ij}=\langle Tv_j, v_i\rangle$$

<u>Def</u> $S^\perp$ for a subset $S\subset V$ (an inner product space)

* A subspace * $\{0\}^\perp=V$ $V^\perp=\{0\}$ * $S\cap S^\perp\subset\{0\}$

<u>Theorem</u> If $W\subset V$ is a subspace & $y\in V$, $\exists!$ $u\in W$ $z\in W^\perp$ s.t. $y=u+z$. If $v_1\dots v_k$ is an O.N basis of W, then $u=\sum_{i=1}^k\langle y,v_i\rangle v_i$

proof of existence was done, but not uniqueness. I said uniqueness will be done either by book or by Kudla.

u is "the orthogonal projection of y on W".

<u>Pf</u> use formula for existence & "sq of var" for uniqueness.

<u>proposition</u> In the situation above, u is the (unique) vector in W which is closest to y:

$$\forall u'\in W,\ \|y-u\|\le\|y-u'\| \text{ equality iff } u=u'.$$

<u>Pf</u> $\|y-u'\|^2=\|u+x-u'\|^2=\|x\|^2+\|u-u'\|^2$

anything below the red line was not touched at all.

<u>Theorem</u> W subspace of a f.d. inner product space V.

1. An ON basis of W can be extended to an ON basis of V

2. $\{w_{k+1}\dots w_n\}$ make an O.N. basis of $W^\perp$

3. $\dim W+\dim W^\perp=\dim V$.

If time: The Riesz representation theorem.

If $g:V\to F$ is a linear transformation (aka "linear functional"), on a f.d. inner prod. sp. V

then $\exists! y \in V$ s.t. $\forall x \in V$ $g(x) = \langle x, y \rangle$

Pf Given an O.N basis $\{v_i\}$, take $y = \sum \overline{g(v_i)} v_i$.

Def/thm $\langle Tx, y \rangle = \langle x, T^* y \rangle$