LimSup Riddle

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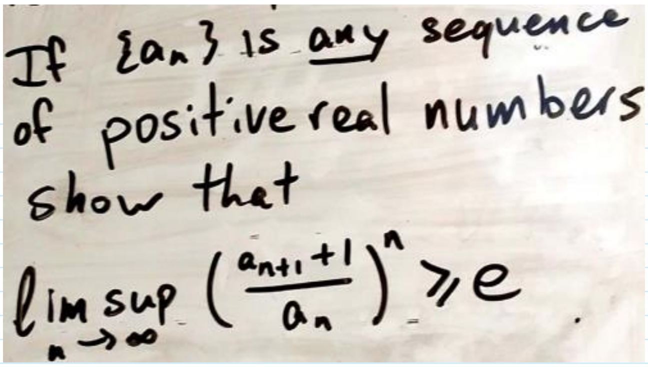

Easily, wlog $a_n \nearrow \infty$.

Note.

$$\log(x+1) \sim \log x + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} \dots$$

$$\lim n(\log(a_{n+1}+1) - \log a_n) = 1$$

$$n\left(\log\frac{a_{n+1}}{a_n} + \frac{1}{a_{n+1}+1}\right)$$

$a_n = n \log n \log^{\circ 2} n \log^{\circ 3} n \dots$ satisfies

$$n\, a_{\log n} = a_n \qquad a_{e^n} = e^n a_n$$

$$\left(\frac{(n+1)\log(n+1)+1}{n\log n}\right)^n \longrightarrow$$

$$\left(\frac{(n+1)(\log n + \frac{1}{n} - \frac{1}{n^2}) + 1}{n\log n}\right)^n$$

$$= \left(\frac{n\log n + \log n + 1 + \frac{1}{n} - \frac{1}{n} + 1}{n\log n}\right)^n = \left(1 + \frac{1}{n} + \frac{2}{n\log n}\right)^n$$

$$\frac{f+f'+1}{f} \sim 1 + \frac{f'}{f} + \frac{1}{f} \qquad \frac{f+f'+\frac{f''}{2}+1}{f}$$

$$\left(\frac{a_{n+1}+1}{a_n}\right)^n < C$$

$$(a_{n+1}+1)^n < C a_n^n$$

$$a_{n+1}+1<c^{1/n}a_n$$

$$a_{n+1}<c^{1/n}a_n-1<c^{\frac{1}{n}+\frac{1}{n-1}}a_{n-1}-c^{1/n}-1$$

$$\dots<c^{\frac{1}{n}+\frac{1}{n-1}+\dots+1}a_0-\sum_{k=0}^{n}c^{\frac{1}{n}+\frac{1}{n-1}+\dots\frac{1}{n-k+1}}$$

$$\sim a_0c^{\log n}-\sum c^{\log n}\left(\left(1+\frac{1}{c}+\frac{1}{c^{1+\frac{1}{2}}}+\frac{1}{c^{1+\frac{1}{2}+\frac{1}{3}}}\right)\right)$$

$$\sum_{k=1}^{n}\frac{1}{c^{\log k}}=\sum_{k=1}^{n}\frac{1}{k^{\log c}}\xrightarrow[n\to\infty]{}\begin{cases}\text{Finite} & c>e\\ \infty & c<e\end{cases}$$