Nikolaev: Abelianisation for sl(2)-Connections

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Questions by DBN: 1. how much is holomorphicity used?
2. would this be interesting on $\mathbb{C}$?
3. on $\mathbb{C}$, could you have given concrete examples for everything?

Let X = cmplx proj. smooth curve
$D :=$ divisor (finite set of pts)

Logarithmic sl_2-connections $(\mathcal{E}, \nabla)$ on (X, D):

$\mathcal{E}$: rk 2 vb on X

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega'_X(D)$ w/ at most simple poles @ D.

residue: if $p \in D$,

$$\mathrm{Res}_p \nabla \in \mathrm{End}(\mathcal{E}|_p)^{\cong sl_2}$$

an invariant of connections, in the sense that if $(\mathcal{E}, \nabla) \cong (\mathcal{E}', \nabla')$ then

$\mathrm{Res}_p \nabla$ & $\mathrm{Res}_p \nabla'$ are in the same adjoint orbit in sl_2

$\Lambda \in sl_2$ is "generic" if

* Λ is regular SS: diagonalizable w/

distinct eigenvalues w/
distinct real parts.

* non-resonance: no two eigvls of Λ should differ by an integer.

An adjoint orbit is generic if it is the orbit of a generic element.

$\mathcal{O}_p$: Adjoint orbit of $\mathrm{Res}_p \nabla$

$\mathcal{O}_D = \{\mathcal{O}_p \mid p \in D\}$ — the residue data.

"$(\mathcal{E}, \nabla)$ has generic residue data" if each $\mathcal{O}_p$ is a generic orbit.

<u>Thm</u> (classical) Suppose $(\mathcal{E}, \nabla)$ is a germ of a logarithmic connection & p. Assume $\mathrm{Res}_p \nabla$ is generic. Then $(\mathcal{E}, \nabla)$ is equipped w/ a natural ∇-invariant filtration $\mathcal{E}_\bullet^p := (0 \subsetneq \mathcal{L}_1^p \subsetneq \mathcal{E})$

(so ∇ restricts to $\mathcal{L}_1^p$)

[The Levelt filtration]

<u>Pf</u> choose a local...

$$\nabla = d - A(x)\frac{dx}{x}$$

$\mathrm{Res}_p \nabla = -A(0)$ $A(0)$ has eigvals

$(\lambda, -\lambda)$ w/ $\mathrm{Re}\,\lambda > 0$

Then use local normal form results..
(Wasow's book, 19(3))

<u>Def'n</u> $(\mathcal{E}, \nabla)$ on (X, D) is "nice" if.

1. $\forall p \in D$ $\mathrm{Res}_p \nabla$ is generic
2. $\forall p, q \in D$ $\mathcal{E}^p_\cdot \pitchfork \mathcal{E}^q_\cdot$. [for each path connecting p & q]

A map between two such connections

$$(\mathcal{E}, \nabla) \xrightarrow{\varphi} (\mathcal{E}', \nabla')$$

is Levelt-filtration preserving if ...

<u>Def'n</u> Fix some generic residue data Θ_D

$$\mathrm{Conn}_X := \left\{ \begin{array}{l} (\mathcal{E}, \nabla) : \text{nice } sl_2 \text{ connections on } (X, D) \\ \text{w/ residue data } \Theta_D, \text{ maps} \\ \text{as above} \end{array} \right\}$$