Pensieve header: Better solving the yaxyax equations.

Representing $\mathfrak{g}^\epsilon$.

```
ME = MatrixExp; MF = MatrixForm;
Simp[sol_] := Flatten[sol] /. ConditionalExpression[ℰ_, _] :> ℰ /.
   (var_ → val_) :> (var → Simplify[PowerExpand@val]);
```

```
ρt = (1 0
      0 1); ρy = (0 0
                  -ϵ 0); ρa = ((1 + 1/ϵ)/2   0
                               0   -(1 - 1/ϵ)/2); ρx = (0 1
                                                        0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}
```

{True, True, True}

```
Simplify[ρt - 2 ϵ ρa] // MF
```

$$\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

```
MF /@ {MatrixFunction[T^# &, ρt], MatrixFunction[A^# &, ρa]}
```

$$\left\{\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} A^{\frac{1+\epsilon}{2\epsilon}} & 0 \\ 0 & A^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} \end{pmatrix}\right\}$$

```
eqn = Simplify@PowerExpand[MatrixFunction[T1^# &, ρt].ME[η1 ρy].MatrixFunction[A1^# &, ρa].
     ME[ξ1 ρx].MatrixFunction[T2^# &, ρt].ME[η2 ρy].MatrixFunction[A2^# &, ρa].ME[ξ2 ρx] ==
    MatrixFunction[T0^# &, ρt].ME[η0 ρy].MatrixFunction[A0^# &, ρa].ME[ξ0 ρx]];
MF /@
 eqn
```

$$\begin{pmatrix} -A1^{\frac{1+\epsilon}{2\epsilon}} A2^{\frac{1+\epsilon}{2\epsilon}} T1\,T2\,(-1+\epsilon\,\eta2\,\xi1) & -A1^{\frac{1+\epsilon}{2\epsilon}} A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} T1\,T2\,(-A2\,\xi2+\xi1\,\ldots \\ A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} A2^{\frac{1+\epsilon}{2\epsilon}} T1\,T2\,\epsilon\,(-\eta2+A1\,\eta1\,(-1+\epsilon\,\eta2\,\xi1)) & A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} T1\,T2\,(1-A2\,\epsilon\,\eta2\,\xi2+A1\,\epsilon\,\eta1\,\ldots \end{pmatrix}$$

```
sol = Solve[Thread[Flatten /@ eqn], {T0, η0, A0, ξ0}, Reals]
```

••• Solve: This system cannot be solved with the methods available to Solve.

$$\mathrm{Solve}\Big[\Big\{-A1^{\frac{1+\epsilon}{2\epsilon}} A2^{\frac{1+\epsilon}{2\epsilon}} T1\,T2\,(-1+\epsilon\,\eta2\,\xi1) == A0^{\frac{1+\epsilon}{2\epsilon}}\,T0,$$
$$-A1^{\frac{1+\epsilon}{2\epsilon}} A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} T1\,T2\,(-A2\,\xi2+\xi1\,(-1+A2\,\epsilon\,\eta2\,\xi2)) == A0^{\frac{1+\epsilon}{2\epsilon}}\,T0\,\xi0,$$
$$A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} A2^{\frac{1+\epsilon}{2\epsilon}} T1\,T2\,\epsilon\,(-\eta2+A1\,\eta1\,(-1+\epsilon\,\eta2\,\xi1)) == -A0^{\frac{1+\epsilon}{2\epsilon}}\,T0\,\epsilon\,\eta0,$$
$$A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} T1\,T2\,(1-A2\,\epsilon\,\eta2\,\xi2+A1\,\epsilon\,\eta1\,(-A2\,\xi2+\xi1\,(-1+A2\,\epsilon\,\eta2\,\xi2))) ==$$
$$A0^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)}\,(T0-A0\,T0\,\epsilon\,\eta0\,\xi0)\Big\}, \{T0, \eta0, A0, \xi0\}, \mathbb{R}\Big]$$

```
MF /@ {MatrixFunction[T^# &, ρt/ϵ], MatrixFunction[A^# &, ρa]}
```

$$\left\{\begin{pmatrix} T^{\frac{1}{\epsilon}} & 0 \\ 0 & T^{\frac{1}{\epsilon}} \end{pmatrix}, \begin{pmatrix} A^{\frac{1+\epsilon}{2\epsilon}} & 0 \\ 0 & A^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)} \end{pmatrix}\right\}$$

```
eqn = Simplify@
   PowerExpand[MatrixFunction[T1# &, ρt / ϵ].ME[η1 ρy].MatrixFunction[A1# &, ρa].ME[ξ1 ρx].
      MatrixFunction[T2# &, ρt / ϵ].ME[η2 ρy].MatrixFunction[A2# &, ρa].ME[ξ2 ρx] ==
     MatrixFunction[T0# &, ρt / ϵ].ME[η0 ρy].MatrixFunction[A0# &, ρa].ME[ξ0 ρx]];
MF /@
 eqn
```

$$\begin{pmatrix} A1^{\frac{1+\epsilon}{2\epsilon}}\, A2^{\frac{1+\epsilon}{2\epsilon}}\, T1^{\frac{1}{\epsilon}}\, T2^{\frac{1}{\epsilon}} \left(1-\epsilon\,\eta 2\,\xi 1\right) & A1^{\frac{1+\epsilon}{2\epsilon}}\, A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)}\, T1^{\frac{1}{\epsilon}}\, T2^{\frac{1}{\epsilon}} \left(\xi 1 + A\ldots\right. \\ A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)}\, A2^{\frac{1+\epsilon}{2\epsilon}}\, T1^{\frac{1}{\epsilon}}\, T2^{\frac{1}{\epsilon}}\, \epsilon \left(-\eta 2 + A1\,\eta 1\left(-1+\epsilon\,\eta 2\,\xi 1\right)\right) & A1^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)}\, A2^{\frac{1}{2}\left(-1+\frac{1}{\epsilon}\right)}\, T1^{\frac{1}{\epsilon}}\, T2^{\frac{1}{\epsilon}} \left(1 - A2\,\epsilon\,\eta 2\,\xi 2 + A1\,\epsilon\ldots\right. \end{pmatrix}$$

```
sol = Solve[Thread[Flatten /@ eqn], {T0, η0, A0, ξ0}]
```

$Aborted

```
eqn = Simplify[ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
    T0 * ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]];
MF /@
 eqn
```

$$\begin{pmatrix} e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}} \left(1-\epsilon\,\eta 2\,\xi 1\right) & e^{\frac{\alpha 1+\alpha 2+\alpha 1\,\epsilon-\alpha 2\,\epsilon}{2\epsilon}} \left(\xi 1 + e^{\alpha 2}\,\xi 2 - e^{\alpha 2}\,\epsilon\,\eta 2\,\xi 1\,\xi 2\right) \\ e^{\frac{\alpha 1+\alpha 2-\alpha 1\,\epsilon+\alpha 2\,\epsilon}{2\epsilon}}\,\epsilon \left(-\eta 2 + e^{\alpha 1}\,\eta 1\left(-1+\epsilon\,\eta 2\,\xi 1\right)\right) & e^{-\frac{(\alpha 1+\alpha 2)(-1+\epsilon)}{2\epsilon}} \left(1 - e^{\alpha 1}\,\epsilon\,\eta 1\,\xi 1 - e^{\alpha 2}\,\epsilon\,\eta 2\,\xi 2 + e^{\alpha 1+\alpha 2}\,\epsilon\,\eta 1 \left(-1+\epsilon\ldots\right.\right. \end{pmatrix}$$

```
sol = Solve[Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}, Reals]
```

$$\Big\{\Big\{T0 \to \text{ConditionalExpression}\Big[-e^{-\frac{(1+\epsilon)\left(\alpha 1+\alpha 2-\text{Log}\left[\frac{1}{(-1+\epsilon\,\eta 2\,\xi 1)^2}\right]\right)}{2\epsilon}} \left(-e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}} + e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}}\,\epsilon\,\eta 2\,\xi 1\right),$$
$$\left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 > 0\right) \,||\, \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 < 0\right)\Big],$$
$$\eta 0 \to \text{ConditionalExpression}\Big[-\left(\left(e^{\alpha 1+\frac{\alpha 1+\alpha 2-\alpha 1\,\epsilon+\alpha 2\,\epsilon}{2\epsilon}}\,\epsilon\,\eta 1 + e^{\frac{\alpha 1+\alpha 2-\alpha 1\,\epsilon+\alpha 2\,\epsilon}{2\epsilon}}\,\epsilon\,\eta 2 - e^{\alpha 1+\frac{\alpha 1+\alpha 2-\alpha 1\,\epsilon+\alpha 2\,\epsilon}{2\epsilon}}\,\epsilon^2\,\eta 1\,\eta 2\,\xi 1\right) \Big/ \right.$$
$$\left.\left(\epsilon\left(-e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}} + e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}}\,\epsilon\,\eta 2\,\xi 1\right)\right)\right), \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 > 0\right) \,||\, \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 < 0\right)\Big],$$
$$\alpha 0 \to \text{ConditionalExpression}\Big[\alpha 1 + \alpha 2 - \text{Log}\Big[\frac{1}{(-1+\epsilon\,\eta 2\,\xi 1)^2}\Big],$$
$$\left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 > 0\right) \,||\, \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 < 0\right)\Big],$$
$$\xi 0 \to \text{ConditionalExpression}\Big[\left(-e^{\frac{\alpha 1+\alpha 2+\alpha 1\,\epsilon-\alpha 2\,\epsilon}{2\epsilon}}\,\xi 1 - e^{\alpha 2+\frac{\alpha 1+\alpha 2+\alpha 1\,\epsilon-\alpha 2\,\epsilon}{2\epsilon}}\,\xi 2 + e^{\alpha 2+\frac{\alpha 1+\alpha 2+\alpha 1\,\epsilon-\alpha 2\,\epsilon}{2\epsilon}}\,\epsilon\,\eta 2\,\xi 1\,\xi 2\right) \Big/$$
$$\left(-e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}} + e^{\frac{(\alpha 1+\alpha 2)(1+\epsilon)}{2\epsilon}}\,\epsilon\,\eta 2\,\xi 1\right), \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 > 0\right) \,||\, \left(\frac{1}{\epsilon\,\eta 2} \in \mathbb{R}\ \&\&\ \eta 2 < 0\right)\Big]\Big\}\Big\}$$

```
Simp[sol]
```

$$\Big\{T0 \to -\left(-1+\epsilon\,\eta 2\,\xi 1\right)^{-1/\epsilon},\ \eta 0 \to \eta 1 + \frac{e^{-\alpha 1}\,\eta 2}{1-\epsilon\,\eta 2\,\xi 1},$$
$$\alpha 0 \to \alpha 1 + \alpha 2 + 2\,\text{Log}[-1+\epsilon\,\eta 2\,\xi 1],\ \xi 0 \to \frac{e^{-\alpha 2}\,\xi 1}{1-\epsilon\,\eta 2\,\xi 1} + \xi 2\Big\}$$

```
FullSimplify[e^(-((α1+α2) (-1+ϵ))/(2 ϵ)) (1 - e^α1 ϵ η1 ξ1 - e^α2 ϵ η2 ξ2 + e^(α1+α2) ϵ η1 (-1 + ϵ η2 ξ1) ξ2)]
```

$$e^{-\frac{(\alpha1+\alpha2)\ (-1+\epsilon)}{2\epsilon}}\left(1-e^{\alpha1}\,\epsilon\,\eta1\,\xi1+e^{\alpha2}\,\epsilon\left(-\eta2+e^{\alpha1}\,\eta1\left(-1+\epsilon\,\eta2\,\xi1\right)\right)\xi2\right)$$

```
ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] // LUDecomposition //
   Simplify // First // MF
```

$$\begin{pmatrix} e^{\frac{(\alpha1+\alpha2)\ (1+\epsilon)}{2\epsilon}}\left(1-\epsilon\,\eta2\,\xi1\right) & e^{\frac{\alpha1+\alpha2+\alpha1\,\epsilon-\alpha2\,\epsilon}{2\epsilon}}\left(\xi1+e^{\alpha2}\,\xi2-e^{\alpha2}\,\epsilon\,\eta2\,\xi1\,\xi2\right) \\ \frac{e^{-\alpha1}\,\epsilon\,\left(\eta2+e^{\alpha1}\,\left(\eta1-\epsilon\,\eta1\,\eta2\,\xi1\right)\right)}{-1+\epsilon\,\eta2\,\xi1} & -\frac{e^{-\frac{(\alpha1+\alpha2)\ (-1+\epsilon)}{2\epsilon}}}{-1+\epsilon\,\eta2\,\xi1} \end{pmatrix}$$

```
Simplify /@ Det /@ eqn
```

$$e^{\frac{\alpha1+\alpha2}{\epsilon}} == e^{\alpha0/\epsilon}\ \mathrm{T0}^2$$

```
sol = Solve[{Simplify /@ Det /@ eqn}~Join~Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]
```

{}

```
M = Table[Which[i > j, 0, i == j, 1, i < j, a_{i,j}], {i, 5}, {j, 5}];
M // MF
```

$$\begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 1 & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & 0 & 1 & a_{3,4} & a_{3,5} \\ 0 & 0 & 0 & 1 & a_{4,5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
MatrixLog[M] // Simplify // MF
```

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3}-\frac{1}{2}a_{1,2}\,a_{2,3} & a_{1,4}+\frac{1}{6}\left(-3\,a_{1,3}\,a_{3,4}+a_{1,2}\left(-3\,a_{2,4}+2\,a_{2,3}\,a_{3,4}\right)\right) & \frac{1}{12}\left(12\,a_{1,5}-6\,a_{1,3}\,a_{3,5}-6\,a\right. \\ 0 & 0 & a_{2,3} & a_{2,4}-\frac{1}{2}a_{2,3}\,a_{3,4} & \\ 0 & 0 & 0 & a_{3,4} & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{pmatrix}$$

```
M = With[{n = 4}, Table[a_{i,j}, {i, n}, {j, n}]];
M // MF
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

```
M // LUDecomposition // First // Simplify // MF
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \\ \frac{a_{2,1}}{a_{1,1}} & -\frac{a_{1,2}\,a_{2,1}}{a_{1,1}}+a_{2,2} & -\frac{a_{1,3}\,a_{2,1}}{a_{1,1}}+a_{2,3} & \\ \frac{a_{3,1}}{a_{1,1}} & \frac{a_{1,2}\,a_{3,1}-a_{1,1}\,a_{3,2}}{a_{1,2}\,a_{2,1}-a_{1,1}\,a_{2,2}} & -\frac{a_{1,3}\,a_{3,1}}{a_{1,1}}+\frac{(a_{1,3}\,a_{2,1}-a_{1,1}\,a_{2,3})\ (-a_{1,2}\,a_{3,1}+a_{1,1}\,a_{3,2})}{a_{1,1}\ (-a_{1,2}\,a_{2,1}+a_{1,1}\,a_{2,2})}+a_{3,3} & \\ \frac{a_{4,1}}{a_{1,1}} & \frac{a_{1,2}\,a_{4,1}-a_{1,1}\,a_{4,2}}{a_{1,2}\,a_{2,1}-a_{1,1}\,a_{2,2}} & \frac{a_{1,3}\ (-a_{2,2}\,a_{4,1}+a_{2,1}\,a_{4,2})+a_{1,2}\ (a_{2,3}\,a_{4,1}-a_{2,1}\,a_{4,3})+a_{1,1}\ (-a_{2,3}\,a_{4,2}+a_{2,2}\,a_{4,3})}{a_{1,3}\ (-a_{2,2}\,a_{3,1}+a_{2,1}\,a_{3,2})+a_{1,2}\ (a_{2,3}\,a_{3,1}-a_{2,1}\,a_{3,3})+a_{1,1}\ (-a_{2,3}\,a_{3,2}+a_{2,2}\,a_{3,3})} & -\frac{a_{1,4}\,a_{4,1}}{a_{1,1}}+\frac{(a_{1,4}\,a_{2,1}-a_{1,1}\,a_{2,4})\ (}{a_{1,1}\ (-a_{1,2}\,a_{2,}} \end{pmatrix}$$