Pensieve header: The $sl_2^{\leq k}$ program.

Some Shortcuts

```
ME[x_] := MatrixExp[x]; MF[x_] := MatrixForm[x];
SeriesOrder[sd_SeriesData] := sd⟦5⟧ - 1;
```

SetDelayed: Tag MatrixExp in MatrixExp[x_] is Protected.

Representing ϵyax

(Borrowed from pensieve://2017-08/Multi-beta-yax.nb and pensieve://2017-09/QuantizedLogos.nb)

$$q = e^{\hbar\gamma\epsilon};\ \rho I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \rho y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix};\ \rho a = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix};$$

$$\rho x = \begin{pmatrix} 0 & \frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar} \\ 0 & 0 \end{pmatrix};\ \rho b = \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix};$$

```
ρt = Simplify[ϵ ρa - γ ρb]; ρA = ME[-ℏ ϵ ρa]; ρB = ME[-ℏ γ ρb]; ρT = ME[ℏ ρt];
(# → MF@Simplify@ToExpression@#) & /@
 {"{{q}}", "ρy", "ρa", "ρx", "ρb", "ρt", "ρA", "ρB", "ρT", "ρI"}
```

$$\left\{\{\{q\}\} \to \left(e^{\gamma\epsilon\hbar}\right),\ \rho y \to \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},\ \rho a \to \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix},\ \rho x \to \begin{pmatrix} 0 & \frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar} \\ 0 & 0 \end{pmatrix},\ \rho b \to \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix},\right.$$

$$\left.\rho t \to \begin{pmatrix} -\gamma\epsilon & 0 \\ 0 & -\gamma\epsilon \end{pmatrix},\ \rho A \to \begin{pmatrix} 1 & 0 \\ 0 & e^{\gamma\epsilon\hbar} \end{pmatrix},\ \rho B \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & 1 \end{pmatrix},\ \rho T \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & e^{-\gamma\epsilon\hbar} \end{pmatrix},\ \rho I \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

```
{ρa.ρx - ρx.ρa == γ ρx, ρx.ρA == q ρA.ρx, ρa.ρy - ρy.ρa == -γ ρy,
  ρb.ρy - ρy.ρb == -ϵ ρy, ρx.ρy - q ρy.ρx == (ρI - ρT.ρA.ρA) / ℏ} // Simplify
```

{True, True, True, True, True}

```
Limit[(ρI - ρT.ρA.ρA) / ℏ, ℏ → 0] == -ρt + 2 ϵ ρa == ρt + 2 γ ρb == γ ρb + ϵ ρa
```

True

The 2-Stitch of Exponentials

```
eqn = (ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx]) ==
    (T0 ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]);
sol = Solve[Thread[Flatten /@ Expand[eqn]], {T0, η0, α0, ξ0}][[1]];
ϕ = sol /. (T0 → ℰ_) :→ (τ0 → Log[ℰ]/(-ϵ γ)) /. (v_ → ℰ_) :→ (v → FullSimplify@PowerExpand[ℰ])
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\tau 0\to \frac{1}{\gamma \epsilon}\left(\gamma \epsilon \left(\tau 1+\tau 2+\hbar\right)+\text{Log}[\hbar]-\text{Log}\left[-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right]\right),\right.$$

$$\eta 0\to \eta 1+\frac{e^{-\alpha 1\,\gamma}\,\eta 2\,\hbar}{\eta 2\left(\xi 1-e^{-\gamma \epsilon \hbar}\,\xi 1\right)+\hbar},$$

$$\alpha 0\to \frac{1}{\gamma}\left(\gamma\left(\alpha 1+\alpha 2-2\,\epsilon\,\hbar\right)-2\,\text{Log}[\hbar]+2\,\text{Log}\left[-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right]\right),$$

$$\left.\xi 0\to \left(e^{-\alpha 2\,\gamma}\left(-e^{\alpha 2\,\gamma}\,\eta 2\,\xi 1\,\xi 2+e^{\gamma \epsilon \hbar}\,\xi 1\,\hbar+e^{\gamma\,(\alpha 2+\epsilon \hbar)}\,\xi 2\left(\eta 2\,\xi 1+\hbar\right)\right)\right)\Big/\left(-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right)\right\}$$

```
Φtot = Simplify[t0 τ0 + y0 η0 + a0 α0 + x0 ξ0 /. ϕ]
```

$$\left(e^{-\alpha 2\,\gamma}\,x0\left(-e^{\alpha 2\,\gamma}\,\eta 2\,\xi 1\,\xi 2+e^{\gamma \epsilon \hbar}\,\xi 1\,\hbar+e^{\gamma\,(\alpha 2+\epsilon \hbar)}\,\xi 2\left(\eta 2\,\xi 1+\hbar\right)\right)\right)\Big/\left(-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right)+$$

$$y0\left(\eta 1+\frac{e^{-\alpha 1\,\gamma}\,\eta 2\,\hbar}{\eta 2\left(\xi 1-e^{-\gamma \epsilon \hbar}\,\xi 1\right)+\hbar}\right)+\frac{1}{\gamma \epsilon}$$

$$t0\left(\gamma \epsilon\left(\tau 1+\tau 2+\hbar\right)+\text{Log}[\hbar]-\text{Log}\left[-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right]\right)+\frac{1}{\gamma}$$

$$a0\left(\gamma\left(\alpha 1+\alpha 2-2\,\epsilon\,\hbar\right)-2\,\text{Log}[\hbar]+2\,\text{Log}\left[-\eta 2\,\xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\,\xi 1+\hbar\right)\right]\right)$$

```
Φ0 = Collect[Limit[Φtot, ϵ → 0], {t0, y0, a0, xξ0}]
```

$$a0\left(\alpha 1+\alpha 2\right)+y0\left(\eta 1+e^{-\alpha 1\,\gamma}\,\eta 2\right)+e^{-\alpha 2\,\gamma}\,x0\,\xi 1+x0\,\xi 2+t0\left(-\eta 2\,\xi 1+\tau 1+\tau 2\right)$$

```
Φi[k_] := Series[Φtot - Φ0, {ϵ, 0, k}];
Φi[2]
```

$$\left(2\,a0\,\eta 2\,\xi 1-e^{-\alpha 1\,\gamma}\,y0\,\gamma\,\eta 2^2\,\xi 1-e^{-\alpha 2\,\gamma}\,x0\,\gamma\,\eta 2\,\xi 1^2+\frac{1}{2}\,t0\,\gamma\,\eta 2^2\,\xi 1^2+\frac{1}{2}\,t0\,\gamma\,\eta 2\,\xi 1\,\hbar\right)\epsilon+$$

$$\left(-a0\,\gamma\,\eta 2^2\,\xi 1^2+e^{-\alpha 1\,\gamma}\,y0\,\gamma^2\,\eta 2^3\,\xi 1^2+e^{-\alpha 2\,\gamma}\,x0\,\gamma^2\,\eta 2^2\,\xi 1^3-\frac{1}{3}\,t0\,\gamma^2\,\eta 2^3\,\xi 1^3-a0\,\gamma\,\eta 2\,\xi 1\,\hbar+\right.$$

$$\left.\frac{1}{2}\,e^{-\alpha 1\,\gamma}\,y0\,\gamma^2\,\eta 2^2\,\xi 1\,\hbar+\frac{1}{2}\,e^{-\alpha 2\,\gamma}\,x0\,\gamma^2\,\eta 2\,\xi 1^2\,\hbar-\frac{1}{2}\,t0\,\gamma^2\,\eta 2^2\,\xi 1^2\,\hbar-\frac{1}{6}\,t0\,\gamma^2\,\eta 2\,\xi 1\,\hbar^2\right)\epsilon^2+O[\epsilon]^3$$

```
Exp[Φi[2]]
```

$$1+\left(2\,a0\,\eta2\,\xi1-e^{-\alpha1\,\gamma}\,y0\,\gamma\,\eta2^2\,\xi1-e^{-\alpha2\,\gamma}\,x0\,\gamma\,\eta2\,\xi1^2+\frac{1}{2}\,t0\,\gamma\,\eta2^2\,\xi1^2+\frac{1}{2}\,t0\,\gamma\,\eta2\,\xi1\,\hbar\right)\epsilon+\frac{1}{2}\left(\left(2\,a0\,\eta2\,\xi1-e^{-\alpha1\,\gamma}\,y0\,\gamma\,\eta2^2\,\xi1-e^{-\alpha2\,\gamma}\,x0\,\gamma\,\eta2\,\xi1^2+\frac{1}{2}\,t0\,\gamma\,\eta2^2\,\xi1^2+\frac{1}{2}\,t0\,\gamma\,\eta2\,\xi1\,\hbar\right)^2+2\left(-a0\,\gamma\,\eta2^2\,\xi1^2+e^{-\alpha1\,\gamma}\,y0\,\gamma^2\,\eta2^3\,\xi1^2+e^{-\alpha2\,\gamma}\,x0\,\gamma^2\,\eta2^2\,\xi1^3-\frac{1}{3}\,t0\,\gamma^2\,\eta2^3\,\xi1^3-a0\,\gamma\,\eta2\,\xi1\,\hbar+\frac{1}{2}\,e^{-\alpha1\,\gamma}\,y0\,\gamma^2\,\eta2^2\,\xi1\,\hbar+\frac{1}{2}\,e^{-\alpha2\,\gamma}\,x0\,\gamma^2\,\eta2\,\xi1^2\,\hbar-\frac{1}{2}\,t0\,\gamma^2\,\eta2^2\,\xi1^2\,\hbar-\frac{1}{6}\,t0\,\gamma^2\,\eta2\,\xi1\,\hbar^2\right)\right)\epsilon^2+O[\epsilon]^3$$

```
Φi[3]
```

$$\left(2\,a0\,\eta2\,\xi1-e^{-\alpha1\,\gamma}\,y0\,\gamma\,\eta2^2\,\xi1-e^{-\alpha2\,\gamma}\,x0\,\gamma\,\eta2\,\xi1^2+\frac{1}{2}\left(t0\,\gamma\,\eta2^2\,\xi1^2+t0\,\gamma\,\eta2\,\xi1\,\hbar\right)\right)\epsilon+\left(-a0\,\gamma\,\eta2^2\,\xi1^2-a0\,\gamma\,\eta2\,\xi1\,\hbar+\frac{1}{2}\,e^{-\alpha1\,\gamma}\,y0\,\gamma^2\,\eta2^2\,\xi1\left(2\,\eta2\,\xi1+\hbar\right)+\frac{1}{2}\,e^{-\alpha2\,\gamma}\,x0\,\gamma^2\,\eta2\,\xi1^2\left(2\,\eta2\,\xi1+\hbar\right)+\frac{1}{6}\left(-2\,t0\,\gamma^2\,\eta2^3\,\xi1^3-3\,t0\,\gamma^2\,\eta2^2\,\xi1^2\,\hbar-t0\,\gamma^2\,\eta2\,\xi1\,\hbar^2\right)\right)\epsilon^2+\left(\frac{2}{3}\,a0\,\gamma^2\,\eta2^3\,\xi1^3+a0\,\gamma^2\,\eta2^2\,\xi1^2\,\hbar+\frac{1}{3}\,a0\,\gamma^2\,\eta2\,\xi1\,\hbar^2-\frac{1}{6}\,e^{-\alpha1\,\gamma}\,y0\,\gamma^3\,\eta2^2\,\xi1\left(6\,\eta2^2\,\xi1^2+6\,\eta2\,\xi1\,\hbar+\hbar^2\right)-\frac{1}{6}\,e^{-\alpha2\,\gamma}\,x0\,\gamma^3\,\eta2\,\xi1^2\left(6\,\eta2^2\,\xi1^2+6\,\eta2\,\xi1\,\hbar+\hbar^2\right)+\frac{1}{24}\left(6\,t0\,\gamma^3\,\eta2^4\,\xi1^4+12\,t0\,\gamma^3\,\eta2^3\,\xi1^3\,\hbar+7\,t0\,\gamma^3\,\eta2^2\,\xi1^2\,\hbar^2+t0\,\gamma^3\,\eta2\,\xi1\,\hbar^3\right)\right)\epsilon^3+O[\epsilon]^4$$

Differential Polynomials (DP)

```
DP_specs:(_→D_)..[P_][f_] := Module[
  {vs = {specs}[[All, 1]] (* variables *),
   dvs = {specs}[[All, 2, 2]] (* dual variables *),
   ps (* powers *), c (* coefficient *)},
  Total[
   CoefficientRules[Normal@P, vs] /.
    (ps_ → c_) :→ c D[f, Sequence @@ Thread[{dvs, ps}]]
  ]
 ]
```

```
DP_{ξ→D_x,η→D_y}[ξ^2 η^3][x^2 y^4]
```

48 y

Stitching

```
m_{i_,j_→k_}[𝔼[ω_, L_, Q_, P_]] := Module[{F, K, ν, δ},
  F = Exp[Φ_i[SeriesOrder@P]];
  K = ν (ξ x + η y + δ x y - t ξ η);

 ]
```