Pensieve header: Reproducing commutation relations from a representation (proves the math wrong!).

Some Shortcuts

```
ME[x_] := MatrixExp[x]; MF[x_] := MatrixForm[x];
```

Representing ϵyax

(Borrowed from pensieve://2017-08/Multi-beta-yax.nb and pensieve://2017-09/QuantizedLogos.nb)

$$q = e^{\hbar\gamma\epsilon};\ \rho I = \begin{pmatrix}1&0\\0&1\end{pmatrix};\ \rho y = \begin{pmatrix}0&0\\-1&0\end{pmatrix};\ \rho a = \begin{pmatrix}0&0\\0&-\gamma\end{pmatrix};$$

$$\rho x = \begin{pmatrix}0&\frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar}\\0&0\end{pmatrix};\ \rho b = \begin{pmatrix}\epsilon&0\\0&0\end{pmatrix};$$

```
ρt = Simplify[ϵ ρa - γ ρb]; ρA = ME[-ℏ ϵ ρa]; ρB = ME[-ℏ γ ρb]; ρT = ME[ℏ ρt];
(# → MF@Simplify@ToExpression@#) & /@
 {"{{q}}", "ρy", "ρa", "ρx", "ρb", "ρt", "ρA", "ρB", "ρT", "ρI"}
```

$$\left\{\{\{q\}\}\to\left(e^{\gamma\epsilon\hbar}\right),\ \rho y\to\begin{pmatrix}0&0\\-1&0\end{pmatrix},\ \rho a\to\begin{pmatrix}0&0\\0&-\gamma\end{pmatrix},\ \rho x\to\begin{pmatrix}0&\frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar}\\0&0\end{pmatrix},\ \rho b\to\begin{pmatrix}\epsilon&0\\0&0\end{pmatrix},\right.$$

$$\left.\rho t\to\begin{pmatrix}-\gamma\epsilon&0\\0&-\gamma\epsilon\end{pmatrix},\ \rho A\to\begin{pmatrix}1&0\\0&e^{\gamma\epsilon\hbar}\end{pmatrix},\ \rho B\to\begin{pmatrix}e^{-\gamma\epsilon\hbar}&0\\0&1\end{pmatrix},\ \rho T\to\begin{pmatrix}e^{-\gamma\epsilon\hbar}&0\\0&e^{-\gamma\epsilon\hbar}\end{pmatrix},\ \rho I\to\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\}$$

```
{ρa.ρx - ρx.ρa == γ ρx, ρx.ρA == q ρA.ρx, ρa.ρy - ρy.ρa == -γ ρy,
  ρb.ρy - ρy.ρb == -ϵ ρy, ρx.ρy - q ρy.ρx == (ρI - ρT.ρA.ρA) / ℏ} // Simplify
```

{True, True, True, True, True}

The 2-Stitch of Exponentials

```
eqn = (ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx]) ==
   (T0 ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]);
sol = Solve[Thread[Flatten /@ Expand[eqn]], {T0, η0, α0, ξ0}][[1]];
ϕ = sol /. (T0 → ℰ_) ⧴ (τ0 → Log[ℰ]/(-ϵ γ)) /.
   (v_ → ℰ_) ⧴ (v → FullSimplify@PowerExpand[ℰ])
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\tau 0\to \frac{1}{\gamma \epsilon}\left(\gamma \epsilon \left(\tau 1+\tau 2+\hbar\right)+\text{Log}[\hbar]-\text{Log}\left[-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right]\right),\right.$$
$$\eta 0\to \eta 1+\frac{e^{-\alpha 1 \gamma}\eta 2\, \hbar}{\eta 2\left(\xi 1-e^{-\gamma \epsilon \hbar}\xi 1\right)+\hbar},$$
$$\alpha 0\to \frac{1}{\gamma}\left(\gamma\left(\alpha 1+\alpha 2-2 \epsilon \hbar\right)-2\,\text{Log}[\hbar]+2\,\text{Log}\left[-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right]\right),$$
$$\left.\xi 0\to \left(e^{-\alpha 2 \gamma}\left(-e^{\alpha 2 \gamma}\eta 2\, \xi 1\, \xi 2+e^{\gamma \epsilon \hbar}\xi 1\, \hbar+e^{\gamma(\alpha 2+\epsilon \hbar)}\xi 2\left(\eta 2\, \xi 1+\hbar\right)\right)\right)\Big/\left(-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right)\right\}$$

```
Φ = Simplify[t0 τ0 + y0 η0 + a0 α0 + x0 ξ0 /. ϕ]
```

$$\left(e^{-\alpha 2 \gamma}x0\left(-e^{\alpha 2 \gamma}\eta 2\, \xi 1\, \xi 2+e^{\gamma \epsilon \hbar}\xi 1\, \hbar+e^{\gamma(\alpha 2+\epsilon \hbar)}\xi 2\left(\eta 2\, \xi 1+\hbar\right)\right)\right)\Big/\left(-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right)+$$
$$y0\left(\eta 1+\frac{e^{-\alpha 1 \gamma}\eta 2\, \hbar}{\eta 2\left(\xi 1-e^{-\gamma \epsilon \hbar}\xi 1\right)+\hbar}\right)+\frac{1}{\gamma \epsilon}$$
$$t0\left(\gamma \epsilon\left(\tau 1+\tau 2+\hbar\right)+\text{Log}[\hbar]-\text{Log}\left[-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right]\right)+\frac{1}{\gamma}$$
$$a0\left(\gamma\left(\alpha 1+\alpha 2-2 \epsilon \hbar\right)-2\,\text{Log}[\hbar]+2\,\text{Log}\left[-\eta 2\, \xi 1+e^{\gamma \epsilon \hbar}\left(\eta 2\, \xi 1+\hbar\right)\right]\right)$$

```
Collect[Limit[Φ, ϵ → 0], {t0, y0, a0, xξ0}]
```

$$a0\left(\alpha 1+\alpha 2\right)+y0\left(\eta 1+e^{-\alpha 1 \gamma}\eta 2\right)+e^{-\alpha 2 \gamma}x0\, \xi 1+x0\, \xi 2+t0\left(-\eta 2\, \xi 1+\tau 1+\tau 2\right)$$

Reproducing the Commutation Relations

```
Simplify@PowerExpand[(∂_{α1,ξ2} e^Φ - ∂_{ξ1,α2} e^Φ) /. {τ1 | η1 | α1 | ξ1 | τ2 | η2 | α2 | ξ2 → 0}]
```

$x0\,\gamma$

```
Simplify@PowerExpand[(∂_{α1,η2} e^Φ - ∂_{η1,α2} e^Φ) /. {τ1 | η1 | α1 | ξ1 | τ2 | η2 | α2 | ξ2 → 0}]
```

$-y0\,\gamma$

```
qbxy = Simplify@PowerExpand[(∂_{ξ1,η2} e^Φ - q ∂_{η1,ξ2} e^Φ) /. {τ1 | η1 | α1 | ξ1 | τ2 | η2 | α2 | ξ2 → 0}]
```

$$-\frac{1}{\gamma \epsilon \hbar}e^{-\gamma \epsilon \hbar}\left(-1+e^{\gamma \epsilon \hbar}\right)\left(t0-2\, a0\, \epsilon+e^{\gamma \epsilon \hbar}x0\, y0\, \gamma \epsilon \hbar\right)$$

```
Limit[qbxy, ϵ → 0]
```

```
-t0
```