Pensieve header: Reproducing commutation relations from a representation, the classical case.

Code borrowed from pensieve://Talks/LesDiablerets-1708/PBWDemo.nb.

Representing $\mathfrak{g}^\epsilon$.

```
ME = MatrixExp;
Simp[sol_] :=
  Flatten[sol] /. C[_] → 0 /. (var_ → val_) :→ (var → Simplify[PowerExpand@val]);
```

$\rho t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\rho y = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix}$; $\rho a = \begin{pmatrix} (1+1/\epsilon)/2 & 0 \\ 0 & -(1-1/\epsilon)/2 \end{pmatrix}$; $\rho x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

```
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}
```

{True, True, True}

The 2-Stitch of Exponentials

$\phi = \left\{\tau_0 \to -\frac{\text{Log}[1 - \epsilon\,\eta_2\,\xi_1]}{\epsilon} + \tau_1 + \tau_2,\ \eta_0 \to \eta_1 + \frac{e^{-\alpha_1}\,\eta_2}{1 - \epsilon\,\eta_2\,\xi_1},\ \alpha_0 \to 2\,\text{Log}[1 - \epsilon\,\eta_2\,\xi_1] + \alpha_1 + \alpha_2,\ \xi_0 \to \frac{e^{-\alpha_2}\,\xi_1}{1 - \epsilon\,\eta_2\,\xi_1} + \xi_2\right\};$

```
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /. ϕ]
```

True

```
Φ = Simplify[t0 τ0 + y0 η0 + a0 α0 + x0 ξ0 /. ϕ]
```

$a_0\,(2\,\text{Log}[1 - \epsilon\,\eta_2\,\xi_1] + \alpha_1 + \alpha_2) + y_0\left(\eta_1 + \frac{e^{-\alpha_1}\,\eta_2}{1 - \epsilon\,\eta_2\,\xi_1}\right) + x_0\left(\frac{e^{-\alpha_2}\,\xi_1}{1 - \epsilon\,\eta_2\,\xi_1} + \xi_2\right) + t_0\left(-\frac{\text{Log}[1 - \epsilon\,\eta_2\,\xi_1]}{\epsilon} + \tau_1 + \tau_2\right)$

```
Collect[Limit[Φ, ϵ → 0], {t0, y0, a0, x0}]
```

$a_0\,(\alpha_1 + \alpha_2) + y_0\,(\eta_1 + e^{-\alpha_1}\,\eta_2) + x_0\,(e^{-\alpha_2}\,\xi_1 + \xi_2) + t_0\,(\eta_2\,\xi_1 + \tau_1 + \tau_2)$

Reproducing the Commutation Relations

```
vp = τ | η | α | ξ;
```

$\text{Simplify@PowerExpand}\left[\left(\partial_{\alpha_1,\xi_2} e^{\Phi} - \partial_{\xi_1,\alpha_2} e^{\Phi}\right) \text{ /. } \{vp\_ \to 0\}\right]$

x_0

```
Simplify@PowerExpand[(∂_{α1,η2} e^Φ - ∂_{η1,α2} e^Φ) /. {vp_ → 0}]
```

$-y_0$

```
bxy = Simplify@PowerExpand[(∂_{ξ1,η2} e^Φ - ∂_{η1,ξ2} e^Φ) /. {vp_ → 0}]
```

$-2\,\epsilon\, a_0 + t_0$

Associativity

```
m_{i_→j_} := {(v : vp)_i ⇸ v_j};
m_{i_→j_,k_} := (ϕ /. {(v : vp)_0 ⇸ v_i, (v : vp)_1 ⇸ v_j, (v : vp)_2 ⇸ v_k});
{m_{2→3}, m_{2→3,4}}
```

$$\left\{\{v\$: \tau \mid \eta \mid \alpha \mid \xi_2 \mapsto v\$_3\},\ \left\{\tau_2 \to -\frac{\text{Log}[1-\epsilon\,\eta_4\,\xi_3]}{\epsilon} + \tau_3 + \tau_4,\ \eta_2 \to \eta_3 + \frac{e^{-\alpha_3}\,\eta_4}{1-\epsilon\,\eta_4\,\xi_3},\ \alpha_2 \to 2\,\text{Log}[1-\epsilon\,\eta_4\,\xi_3] + \alpha_3 + \alpha_4,\ \xi_2 \to \frac{e^{-\alpha_4}\,\xi_3}{1-\epsilon\,\eta_4\,\xi_3} + \xi_4\right\}\right\}$$

```
lhs = Simplify[Φ /. m_{2→2,3}]
```

$$a_0\left(2\,\text{Log}[1-\epsilon\,\eta_3\,\xi_2] + 2\,\text{Log}\left[1-\epsilon\,\xi_1\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)\right] + \alpha_1+\alpha_2+\alpha_3\right) +$$
$$y_0\left(\eta_1 + \frac{e^{-\alpha_1}\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)}{1-\epsilon\,\xi_1\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)}\right) + x_0\left(\frac{e^{-\alpha_3}\,\xi_2}{1-\epsilon\,\eta_3\,\xi_2} + \frac{e^{-\alpha_2-\alpha_3}\,\xi_1}{(-1+\epsilon\,\eta_3\,\xi_2)^2\left(1-\epsilon\,\xi_1\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)\right)} + \xi_3\right) +$$
$$t_0\left(\tau_1+\tau_2-\frac{1}{\epsilon}\left(\text{Log}[1-\epsilon\,\eta_3\,\xi_2] + \text{Log}\left[1-\epsilon\,\xi_1\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)\right] - \epsilon\,\tau_3\right)\right)$$

```
rhs = Simplify[Φ /. m_{2→3} /. m_{1→1,2}]
```

$$a_0\left(2\,\text{Log}[1-\epsilon\,\eta_2\,\xi_1] + 2\,\text{Log}\left[1-\epsilon\,\eta_3\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)\right] + \alpha_1+\alpha_2+\alpha_3\right) +$$
$$y_0\left(\eta_1 + \left(e^{-\alpha_1}\left(\eta_3 - e^{\alpha_2}\,\eta_2\,(-1+\epsilon\,\eta_3\,\xi_2)\right)\right) \Big/ \left(e^{\alpha_2} - \epsilon\,\eta_3\,(\xi_1 + e^{\alpha_2}\,\xi_2) + e^{\alpha_2}\,\epsilon\,\eta_2\,\xi_1\,(-1+\epsilon\,\eta_3\,\xi_2)\right)\right) +$$
$$x_0\left(\frac{e^{-\alpha_3}\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)}{1-\epsilon\,\eta_3\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)} + \xi_3\right) +$$
$$t_0\left(\tau_1+\tau_2-\frac{1}{\epsilon}\left(\text{Log}[1-\epsilon\,\eta_2\,\xi_1] + \text{Log}\left[1-\epsilon\,\eta_3\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)\right] - \epsilon\,\tau_3\right)\right)$$

```
FullSimplify[lhs == rhs]
```

$$\frac{1}{\epsilon}\left(\text{Log}[1-\epsilon\,\eta_2\,\xi_1] - \text{Log}[1-\epsilon\,\eta_3\,\xi_2] + \text{Log}\left[1-\epsilon\,\eta_3\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)\right] - \text{Log}\left[1-\epsilon\,\xi_1\left(\eta_2+\frac{e^{-\alpha_2}\,\eta_3}{1-\epsilon\,\eta_3\,\xi_2}\right)\right]\right)(2\,\epsilon\,a_0 - t_0) = 0$$

```
Simplify[
 FullSimplify[lhs == rhs] //. {Log[a_] + Log[b_] :> Log[a b], Log[a_] - Log[b_] :> Log[a / b]}
]
```

```
True
```