Pensieve header: The sl_2^ϵ dequantizator.

Code borrowed from Phi2CR-Classical.nb.

Representing $\mathfrak{g}^\epsilon$.

```
ME = MatrixExp;
Simp[sol_] :=
  Flatten[sol] /. C[_] → 0 /. (var_ → val_) :> (var → Simplify[PowerExpand@val]);
```

$\rho t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\rho y = \begin{pmatrix} 0 & 0 \\ -\epsilon & 0 \end{pmatrix}$; $\rho a = \begin{pmatrix} (1+1/\epsilon)/2 & 0 \\ 0 & -(1-1/\epsilon)/2 \end{pmatrix}$; $\rho x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;

```
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}
{True, True, True}
```

The 2-Stitch of Exponentials

$$\phi = \left\{\tau_0 \to -\frac{\text{Log}[1 - \epsilon\,\eta_2\,\xi_1]}{\epsilon} + \tau_1 + \tau_2,\ \eta_0 \to \eta_1 + \frac{e^{-\alpha_1}\,\eta_2}{1 - \epsilon\,\eta_2\,\xi_1},\ \alpha_0 \to 2\,\text{Log}[1 - \epsilon\,\eta_2\,\xi_1] + \alpha_1 + \alpha_2,\ \xi_0 \to \frac{e^{-\alpha_2}\,\xi_1}{1 - \epsilon\,\eta_2\,\xi_1} + \xi_2\right\};$$

```
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /. ϕ]
True
```

```
Φ = Simplify[t0 τ0 + y0 η0 + a0 α0 + x0 ξ0 /. ϕ]
```

$$a_0\,(2\,\text{Log}[1 - \epsilon\,\eta_2\,\xi_1] + \alpha_1 + \alpha_2) + y_0\left(\eta_1 + \frac{e^{-\alpha_1}\,\eta_2}{1 - \epsilon\,\eta_2\,\xi_1}\right) + x_0\left(\frac{e^{-\alpha_2}\,\xi_1}{1 - \epsilon\,\eta_2\,\xi_1} + \xi_2\right) + t_0\left(-\frac{\text{Log}[1 - \epsilon\,\eta_2\,\xi_1]}{\epsilon} + \tau_1 + \tau_2\right)$$

```
Collect[Limit[Φ, ϵ → 0], {t0, y0, a0, x0}]
```

$$a_0\,(\alpha_1 + \alpha_2) + y_0\,(\eta_1 + e^{-\alpha_1}\,\eta_2) + x_0\,(e^{-\alpha_2}\,\xi_1 + \xi_2) + t_0\,(\eta_2\,\xi_1 + \tau_1 + \tau_2)$$

Finding the Dequantizator

Should satisfy $x\,\hat{y} - q\,\hat{y}\,x = \frac{1 - e^{\hbar(t - 2\epsilon a)}}{\hbar}$ where $q = e^{\hbar\gamma\epsilon}$ (temporarily at $\gamma = 1$).

The $x\,\hat{y}$ side

```
Simplify[∂_{ξ1} e^Φ]
```

$$\frac{1}{\left(-1+\epsilon\,\eta_2\,\xi_1\right)^2}\,e^{-\alpha_1-\alpha_2+a_0\,\left(2\,\mathrm{Log}[1-\epsilon\,\eta_2\,\xi_1]+\alpha_1+\alpha_2\right)+y_0\,\left(\eta_1+\frac{e^{-\alpha_1}\,\eta_2}{1-\epsilon\,\eta_2\,\xi_1}\right)+x_0\,\left(\frac{e^{-\alpha_2}\,\xi_1}{1-\epsilon\,\eta_2\,\xi_1}+\xi_2\right)+t_0\,\left(-\frac{\mathrm{Log}[1-\epsilon\,\eta_2\,\xi_1]}{\epsilon}+\tau_1+\tau_2\right)}$$
$$\left(e^{\alpha_1}\,x_0+e^{\alpha_2}\,\eta_2\,\left(\epsilon\,y_0\,\eta_2+2\,e^{\alpha_1}\,\epsilon\,a_0\,\left(-1+\epsilon\,\eta_2\,\xi_1\right)-e^{\alpha_1}\,t_0\,\left(-1+\epsilon\,\eta_2\,\xi_1\right)\right)\right)$$

```
Simplify[∂_{ξ1} e^Φ] /. {(τ | η | α | ξ)_1 → 0, (ν : (τ | η | α | ξ))_2 :→ ν, (ν : (t | y | a | x))_0 :→ ν}
```

$$e^{-\alpha+a\,\alpha+y\,\eta+x\,\xi+t\,\tau}\,\left(x+e^{\alpha}\,\eta\,\left(t-2\,a\,\epsilon+y\,\epsilon\,\eta\right)\right)$$

Makes $x\,f[y,\,a-1]+(t-2\,a\,\epsilon+y\,\epsilon\,\partial_y)\,\partial_y\,f[y,\,a]$.

The $q\,\hat{y}\,x$ side

```
Simplify[∂_{ξ2} e^Φ] /. {(ν : (τ | η | α | ξ))_1 :→ ν, (τ | η | α | ξ)_2 → 0, (ν : (t | y | a | x))_0 :→ ν}
```

$$e^{a\,\alpha+y\,\eta+x\,\xi+t\,\tau}\,x$$

Makes $x\,e^{\hbar\,\epsilon}\,f[y,\,a]$.

The overall equation

$x\left(f[y,\,a-1]-e^{\hbar\,\epsilon}\,f[y,\,a]\right)+(t-2\,a\,\epsilon+y\,\epsilon\,\partial_y)\,\partial_y\,f[y,\,a]=\frac{1-e^{\hbar(t-2\,\epsilon\,a)}}{\hbar}$.

At $\epsilon=0$ this is $x\,(f[y,\,a-1]-f[y,\,a])+t\,\partial_y\,f[y,\,a]=\frac{1-e^{\hbar t}}{\hbar}$, which is solved by $f[y,\,a]=y\,\frac{1-e^{\hbar t}}{\hbar\,t}$.

With $f[y,\,a]=e^{-\hbar\,\epsilon\,a}\,g[y]$ the equation becomes $(t-2\,a\,\epsilon+y\,\epsilon\,\partial_y)\,\partial_y\,g[y]=\frac{e^{\hbar\,\epsilon\,a}-e^{\hbar(t-\epsilon\,a)}}{\hbar}$.

```
{sol} = DSolve[(t - 2 a ϵ) g'[y] + y ϵ g''[y] == (e^(ħ ϵ a) - e^(ħ (t - ϵ a)))/ħ, g[y], y]
```

$$\left\{\left\{g[y]\to\left(e^{a\,\epsilon\,\hbar}\,y-e^{(t-a\,\epsilon)\,\hbar}\,y+\frac{y^{1+2\,a-\frac{t}{\epsilon}}\,\left(t-2\,a\,\epsilon\right)\,\hbar\,C[1]}{1+2\,a-\frac{t}{\epsilon}}\right)\Big/\left(\left(t-2\,a\,\epsilon\right)\,\hbar\right)+C[2]\right\}\right\}$$

```
Simplify[g[y] /. sol /. C[_] → 0]
```

$$\frac{\left(e^{a\,\epsilon\,\hbar}-e^{(t-a\,\epsilon)\,\hbar}\right)\,y}{\left(t-2\,a\,\epsilon\right)\,\hbar}$$