Pensieve header: Δ and S in the quantized case; evolved from pensieve://2017-09/nb/QuantizedLogos.pdf.

Some Shortcuts

```
ME[x_] := MatrixExp[x]; MF[x_] := MatrixForm[x];
```

Representing ϵyax

(Borrowed from pensieve://2017-08/Multi-beta-yax.nb)

```
q = e^(ħ γ ϵ); ρI = (1 0; 0 1); ρy = (0 0; -1 0); ρa = (0 0; 0 -γ);
ρx = (0 (-1+e^(-γ ϵ ħ))/ħ; 0 0); ρb = (ϵ 0; 0 0);
ρt = Simplify[ϵ ρa - γ ρb]; ρA = ME[-ħ ϵ ρa]; ρB = ME[-ħ γ ρb]; ρT = ME[ħ ρt];
(# → MF@Simplify@ToExpression@#) & /@
 {"{{q}}", "ρy", "ρa", "ρx", "ρb", "ρt", "ρA", "ρB", "ρT", "ρI"}
```

$\{\{\{q\}\} \to \begin{pmatrix} e^{\gamma\epsilon\hbar}\end{pmatrix}, \rho y \to \begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix}, \rho a \to \begin{pmatrix} 0 & 0 \\ 0 & -\gamma\end{pmatrix}, \rho x \to \begin{pmatrix} 0 & \frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar} \\ 0 & 0\end{pmatrix}, \rho b \to \begin{pmatrix} \epsilon & 0 \\ 0 & 0\end{pmatrix},$
$\rho t \to \begin{pmatrix} -\gamma\epsilon & 0 \\ 0 & -\gamma\epsilon\end{pmatrix}, \rho A \to \begin{pmatrix} 1 & 0 \\ 0 & e^{\gamma\epsilon\hbar}\end{pmatrix}, \rho B \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & 1\end{pmatrix}, \rho T \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & e^{-\gamma\epsilon\hbar}\end{pmatrix}, \rho I \to \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\}$

```
{ρa.ρx - ρx.ρa == γ ρx, ρx.ρA == q ρA.ρx, ρa.ρy - ρy.ρa == -γ ρy,
  ρb.ρy - ρy.ρb == -ϵ ρy, ρx.ρy - q ρy.ρx == (ρI - ρT.ρA.ρA) / ħ} // Simplify
```

{True, True, True, True, True}

Δ(x)

Note that $\Delta(x) = x \otimes 1 + A \otimes x$. We can regard the second tensor factor as commuting coefficients.

```
MF /@ (eqn = ME[ξ (ρx + x ρA)] == ME[α0 ρA].ME[ξ0 ρx])
```

$\begin{pmatrix} e^{x\xi} & \frac{e^{-\gamma\epsilon\hbar}\left(e^{x\xi}-e^{e^{\gamma\epsilon\hbar}x\xi}\right)}{x\hbar} \\ 0 & e^{e^{\gamma\epsilon\hbar}x\xi}\end{pmatrix} == \begin{pmatrix} e^{\alpha0} & -\frac{e^{\alpha0-\gamma\epsilon\hbar}\left(-1+e^{\gamma\epsilon\hbar}\right)\xi0}{\hbar} \\ 0 & e^{e^{\gamma\epsilon\hbar}\alpha0}\end{pmatrix}$

```
Thread[Flatten /@ eqn /. α0 → ξ x]
```

$\{\text{True}, \frac{e^{-\gamma\epsilon\hbar}\left(e^{x\xi}-e^{e^{\gamma\epsilon\hbar}x\xi}\right)}{x\hbar} == -\frac{e^{x\xi-\gamma\epsilon\hbar}\left(-1+e^{\gamma\epsilon\hbar}\right)\xi0}{\hbar}, \text{True}, \text{True}\}$

```
sol = Solve[Thread[Flatten /@ eqn /. α0 → ξ x], {ξ0}][[1]]
```

$\{\xi0 \to -\frac{e^{-x\xi}\left(e^{x\xi}-e^{e^{\gamma\epsilon\hbar}x\xi}\right)}{\left(-1+e^{\gamma\epsilon\hbar}\right)x}\}$

```
FullSimplify[ξ0 /. sol]
```

$$\frac{-1+e^{\left(-1+e^{\gamma \epsilon \hbar}\right) x \xi}}{\left(-1+e^{\gamma \epsilon \hbar}\right) x}$$

```
Series[ξ0 /. sol, {ϵ, 0, 3}]
```

$$\xi+\frac{1}{2} x \gamma \xi^2 \hbar \epsilon+\frac{1}{12} x\left(3 \gamma^2 \xi^2 \hbar^2+2 x \gamma^2 \xi^3 \hbar^2\right) \epsilon^2+\frac{1}{24} x\left(2 \gamma^3 \xi^2 \hbar^3+4 x \gamma^3 \xi^3 \hbar^3+x^2 \gamma^3 \xi^4 \hbar^3\right) \epsilon^3+O[\epsilon]^4$$

$S(x)$

Note that $S(x) = -A^{-1}x$.

```
-Inverse[ρA].ρx // Simplify // MF
```

$$\begin{pmatrix} 0 & \frac{1-e^{-\gamma \epsilon \hbar}}{\hbar} \\ 0 & 0 \end{pmatrix}$$

```
MF /@ (eqn = ME[-ξ Inverse[ρA].ρx] == ME[ξ0 ρx])
```

$$\begin{pmatrix} 1 & \frac{e^{-\gamma \epsilon \hbar}\left(-1+e^{\gamma \epsilon \hbar}\right) \xi}{\hbar} \\ 0 & 1 \end{pmatrix} == \begin{pmatrix} 1 & -\frac{e^{-\gamma \epsilon \hbar}\left(-1+e^{\gamma \epsilon \hbar}\right) \xi 0}{\hbar} \\ 0 & 1 \end{pmatrix}$$

```
Thread[Flatten /@ eqn]
```

$$\left\{\text{True}, \frac{e^{-\gamma \epsilon \hbar}\left(-1+e^{\gamma \epsilon \hbar}\right) \xi}{\hbar} == -\frac{e^{-\gamma \epsilon \hbar}\left(-1+e^{\gamma \epsilon \hbar}\right) \xi 0}{\hbar}, \text{True}, \text{True}\right\}$$

```
sol = Solve[Thread[Flatten /@ eqn], {ξ0}][[1]]
```

$\{\xi 0 \to -\xi\}$

```
FullSimplify[ξ0 /. sol]
```

$-\xi$

```
Series[ξ0 /. sol, {ϵ, 0, 3}]
```

$-\xi$

$\Delta(y)$

Note that $\Delta(y) = y \otimes B + 1 \otimes y$. We can regard the first tensor factor as commuting coefficients.

```
MF /@ (eqn = ME[η (y ρB + ρy)] == ME[η0 ρy].ME[β0 ρB])
```

$$\begin{pmatrix} e^{e^{-\gamma \epsilon \hbar} y \eta} & 0 \\ -\frac{e^{\gamma \epsilon \hbar}\left(e^{y \eta}-e^{e^{-\gamma \epsilon \hbar} y \eta}\right)}{\left(-1+e^{\gamma \epsilon \hbar}\right) y} & e^{y \eta} \end{pmatrix} == \begin{pmatrix} e^{e^{-\gamma \epsilon \hbar} \beta 0} & 0 \\ -e^{e^{-\gamma \epsilon \hbar} \beta 0} \eta 0 & e^{\beta 0} \end{pmatrix}$$

```
Thread[Flatten /@ eqn /. β0 → η y]
```

$$\left\{\text{True}, \text{True}, -\frac{e^{\gamma \in \hbar}\left(e^{y\,\eta}-e^{e^{-\gamma \in \hbar}\,y\,\eta}\right)}{\left(-1+e^{\gamma \in \hbar}\right)\,y} == -e^{e^{-\gamma \in \hbar}\,y\,\eta}\,\eta 0, \text{True}\right\}$$

```
sol = Solve[Thread[Flatten /@ eqn /. β0 → η y], {η0}]〚1〛
```

$$\left\{\eta 0 \to -\frac{e^{-e^{-\gamma \in \hbar}\,y\,\eta+\gamma \in \hbar}\left(-e^{y\,\eta}+e^{e^{-\gamma \in \hbar}\,y\,\eta}\right)}{\left(-1+e^{\gamma \in \hbar}\right)\,y}\right\}$$

```
FullSimplify[η0 /. sol]
```

$$\frac{e^{\gamma \in \hbar}\left(-1+e^{\left(1-e^{-\gamma \in \hbar}\right)\,y\,\eta}\right)}{\left(-1+e^{\gamma \in \hbar}\right)\,y}$$

```
Series[η0 /. sol, {ϵ, 0, 3}]
```

$$\eta+\frac{1}{2}\,y\,\gamma\,\eta^2\,\hbar \in +\frac{1}{12}\,y\left(-3\,\gamma^2\,\eta^2\,\hbar^2+2\,y\,\gamma^2\,\eta^3\,\hbar^2\right)\in^2+$$
$$\frac{1}{24}\,y\left(2\,\gamma^3\,\eta^2\,\hbar^3-4\,y\,\gamma^3\,\eta^3\,\hbar^3+y^2\,\gamma^3\,\eta^4\,\hbar^3\right)\in^3+O[\in]^4$$