Pensieve header: Formulas for the Cartan involution (failed).

Code borrowed from Dequantizator.nb.

Monoblog on Oct 31, 2017:

(170625) $\mathcal{U}_{\hbar;\gamma\epsilon}$ conventions: $q=\mathbb{e}^{\hbar\gamma\epsilon}$, $H=\langle a,x\rangle/([a,x]=\gamma x)$ with

$$A=\mathbb{e}^{-\hbar\epsilon a},\quad xA=qAx,\quad S(a,A,x)=(-a,A^{-1},-A^{-1}x),$$
$$\Delta(a,A,x)=(a_1+a_2,A_1A_2,x_1+A_1x_2)$$

and dual $H^*=\langle b,y\rangle/([b,y]=-\epsilon y)$ with

$$B=\mathbb{e}^{-\hbar\gamma b},\quad By=qyB,\quad S(b,B,y)=(-b,B^{-1},-yB^{-1}),$$
$$\Delta(b,B,y)=(b_1+b_2,B_1B_2,y_1B_2+y_2).$$

Pairing by $(a,x)^*=\hbar(b,y)$ making $\langle y^lb^i,a^jx^k\rangle=\delta_{ij}\delta_{kl}\hbar^{-(j+k)}j![k]_q!$ so $R=\sum\frac{\hbar^{j+k}y^kb^j\otimes a^jx^k}{j![k]_q!}$. Then $\mathcal{U}=H^{*cop}\otimes H$ with $(\phi f)(\psi g)=\langle\psi_1S^{-1}f_3\rangle\langle\psi_3,f_1\rangle(\phi\psi_2)(f_2g)$. With the central $t:=\epsilon a-\gamma b$, $T:=e^{\hbar t}=A^{-1}B$ get

$$[a,y]=-\gamma y,\quad [b,x]=\epsilon x,\quad xy-qyx=(1-TA^2)/\hbar.$$

Cartan: $\theta(y,b,a,x)=(-B^{-1}T^{1/2}x,-b,-a,-A^{-1}T^{-1/2}y)$.
At $\epsilon=0$, $\mathcal{U}_{\hbar;\gamma 0}=\langle b,y,a,x\rangle/([b,\cdot]=0,[a,x]=\gamma x,[a,y]=-\gamma y,[x,y]=(1-\mathbb{e}^{-\hbar\gamma b})/\hbar)$ with $\Delta(b,y,a,x)=(b_1+b_2,y_1+\mathbb{e}^{-\hbar\gamma b_1}y_2,a_1+a_2,x_1+x_2)$ and $\theta(y,b,a,x)=(-\mathbb{e}^{\hbar\gamma b/2}x,-b,-a,-\mathbb{e}^{\hbar\gamma b/2}y)$.

Representing $\mathfrak{g}^\epsilon$.

```
ME = MatrixExp; MF = MatrixForm;
Simp[sol_] := sol /. {ss_List} :> ss /. C[_] → 0 /.
    (var_ → val_) :> (var → Simplify[PowerExpand@val]);
```

$$\rho t=\begin{pmatrix}1&0\\0&1\end{pmatrix};\ \rho y=\begin{pmatrix}0&0\\\epsilon&0\end{pmatrix};\ \rho a=\begin{pmatrix}(1+1/\epsilon)/2&0\\0&-(1-1/\epsilon)/2\end{pmatrix};\ \rho x=\begin{pmatrix}0&1\\0&0\end{pmatrix};$$

```
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy, ρx.ρy - ρy.ρx == 2 ϵ ρa - ρt}
```

```
{True, True, True}
```

Finding θ

At $\hbar=0$, we have $\theta(t,y,a,x)=(-t,-x,-a,-y)$.

```
eqn = Simplify[ME[-τ1 ρt].ME[-η1 ρx].ME[-α1 ρa].ME[-ξ1 ρy] ==
    ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]];
MF /@
 eqn
```

$$\begin{pmatrix}\mathbb{e}^{-\frac{\alpha1+\alpha1\,\epsilon+2\,\epsilon\,\tau1}{2\,\epsilon}}\left(1+\mathbb{e}^{\alpha1}\,\epsilon\,\eta1\,\xi1\right) & -\mathbb{e}^{\frac{\alpha1\,(-1+\epsilon)}{2\,\epsilon}-\tau1}\,\eta1\\ -\mathbb{e}^{\frac{\alpha1\,(-1+\epsilon)}{2\,\epsilon}-\tau1}\,\epsilon\,\xi1 & \mathbb{e}^{\frac{\alpha1\,(-1+\epsilon)}{2\,\epsilon}-\tau1}\end{pmatrix} == \begin{pmatrix}\mathbb{e}^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}+\tau0} & \mathbb{e}^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}+\tau0}\,\xi0\\ \mathbb{e}^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}+\tau0}\,\epsilon\,\eta0 & \mathbb{e}^{-\frac{\alpha0\,(-1+\epsilon)}{2\,\epsilon}+\tau0}\left(1+\mathbb{e}^{\alpha0}\,\epsilon\,\eta0\,\xi0\right)\end{pmatrix}$$

```
sol = Solve[Thread[Flatten /@ eqn], {τ0, η0, α0, ξ0}]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$\epsilon\left(-2\,\mathrm{Log}\left[e^{\frac{\alpha0+\mathrm{Times}[\ll 2\gg]+\mathrm{Times}[\ll 3\gg]}{\epsilon}}\right]+4\,\mathrm{Log}\left[e^{-\frac{1}{2}\mathrm{Power}[\ll 2\gg]\,\mathrm{Plus}[\ll 3\gg]}\,(1+\mathrm{Power}[\ll 2\gg]\,\epsilon\,\eta1\,\xi1)\right]\right)==0.$$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$\epsilon\left(4\,\tau1-2\,\mathrm{Log}\left[e^{\frac{\alpha0+\mathrm{Times}[\ll 2\gg]+\ll 1\gg+\mathrm{Times}[\ll 3\gg]}{\epsilon}}\right]+4\,\mathrm{Log}\left[e^{-\frac{1}{2}\mathrm{Power}[\ll 2\gg]\,\mathrm{Plus}[\ll 3\gg]}\,(1+\mathrm{Power}[\ll 2\gg]\,\epsilon\,\eta1\,\xi1)\right]\right)==0.$$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$-2\,\tau1+\mathrm{Log}\left[e^{\frac{\alpha0+\alpha0\,\epsilon+2\,\epsilon\,\tau0+2\,\epsilon\,\tau1}{\epsilon}}\right]-2\,\mathrm{Log}\left[e^{-\frac{\alpha1+\mathrm{Times}[\ll 2\gg]+\mathrm{Times}[\ll 3\gg]}{2\epsilon}}\,(1+e^{\alpha1}\,\epsilon\,\eta1\,\xi1)\right]==0.$$

General: Further output of Solve::incnst will be suppressed during this calculation.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{\tau0\to\frac{1}{2}\left(-\mathrm{Log}\left[e^{\alpha0/\epsilon}\right]-\epsilon\,\mathrm{Log}\left[e^{\alpha0/\epsilon}\right]+2\,\mathrm{Log}\left[e^{-\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}+e^{\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}\,\epsilon\,\eta1\,\xi1\right]\right),\right.$$

$$\left.\eta0\to e^{\frac{-\alpha1+2\,i\,\pi\,\epsilon+\alpha1\,\epsilon-2\,\epsilon\,\tau1+2\,\epsilon\,\mathrm{Log}[\xi1]-2\,\epsilon\,\mathrm{Log}\left[e^{-\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}+e^{\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}\,\epsilon\,\eta1\,\xi1\right]}{2\epsilon}},\ \xi0\to e^{\frac{-\alpha1-2\,i\,\pi\,\epsilon+\alpha1\,\epsilon-2\,\epsilon\,\tau1+2\,\epsilon\,\mathrm{Log}[\eta1]-2\,\epsilon\,\mathrm{Log}\left[e^{-\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}+e^{\frac{\alpha1}{2}-\frac{\alpha1}{2\epsilon}-\tau1}\,\epsilon\,\eta1\,\xi1\right]}{2\epsilon}}\right\}\right\}$$

```
eqn = Simplify[ME[-η1 ρx].ME[0 ρa].ME[-ξ1 ρy] == T0 * ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]];
MF /@ eqn
sol = Simp@Solve[eqn1 = Thread[Flatten /@ eqn], {T0, η0, α0, ξ0}]
```

$$\begin{pmatrix}1+\epsilon\,\eta1\,\xi1 & -\eta1\\ -\epsilon\,\xi1 & 1\end{pmatrix}==\begin{pmatrix}e^{\frac{\alpha0\,(1+\epsilon)}{2\epsilon}}\,T0 & e^{\frac{\alpha0\,(1+\epsilon)}{2\epsilon}}\,T0\,\xi0\\ e^{\frac{\alpha0\,(1+\epsilon)}{2\epsilon}}\,T0\,\epsilon\,\eta0 & e^{-\frac{\alpha0\,(-1+\epsilon)}{2\epsilon}}\,T0\,\left(1+e^{\alpha0}\,\epsilon\,\eta0\,\xi0\right)\end{pmatrix}$$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{T0\to e^{-\frac{\alpha0\,(1+\epsilon)}{2\epsilon}}\,(1+\epsilon\,\eta1\,\xi1),\ \eta0\to-\frac{1+e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}}{2\,\epsilon\,\eta1},\ \xi0\to-\frac{1+e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}}{2\,\epsilon\,\xi1}\right\},\right.$$

$$\left.\left\{T0\to e^{-\frac{\alpha0\,(1+\epsilon)}{2\epsilon}}\,(1+\epsilon\,\eta1\,\xi1),\ \eta0\to-\frac{1-e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}}{2\,\epsilon\,\eta1},\ \xi0\to-\frac{1-e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}}{2\,\epsilon\,\xi1}\right\}\right\}$$

```
FullSimplify[eqn1 /. sol]
```

$$\left\{\left\{\text{True}, \frac{1}{2\,\epsilon\,\xi1}(1+\epsilon\,\eta1\,\xi1)\left(1+e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}\right)==\eta1,\right.\right.$$
$$\frac{1}{2\,\eta1}(1+\epsilon\,\eta1\,\xi1)\left(1+e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}\right)==\epsilon\,\xi1,$$
$$\left.\frac{1}{2\,\epsilon\,\eta1\,\xi1}e^{-\alpha0/2}(1+\epsilon\,\eta1\,\xi1)\left(e^{\alpha0/2}+\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}\right)==1\right\},$$
$$\left\{\text{True}, \frac{1}{2\,\epsilon\,\xi1}(1+\epsilon\,\eta1\,\xi1)\left(1-e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}\right)==\eta1,\right.$$
$$\left.\left.\frac{1}{\eta1}\left(-1+\epsilon\,\eta1\,\xi1+e^{-\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}\,(1+\epsilon\,\eta1\,\xi1)\right)==0,\ \frac{2+2\,\epsilon\,\eta1\,\xi1}{e^{\alpha0}+e^{\alpha0/2}\sqrt{e^{\alpha0}-4\,\epsilon\,\eta1\,\xi1}}==1\right\}\right\}$$

```
eqn = Simplify[ME[-η1 ρx].ME[0 ρa].ME[-ξ1 ρy] == ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx]];
MF /@ eqn
sol = Simp@Solve[eqn1 = Thread[Flatten /@ eqn], {η0, α0, ξ0}]
```

$$\begin{pmatrix} 1+\epsilon\,\eta1\,\xi1 & -\eta1 \\ -\epsilon\,\xi1 & 1 \end{pmatrix} == \begin{pmatrix} e^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}} & e^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}}\,\xi0 \\ e^{\frac{\alpha0\,(1+\epsilon)}{2\,\epsilon}}\,\epsilon\,\eta0 & e^{-\frac{\alpha0\,(-1+\epsilon)}{2\,\epsilon}}\left(1+e^{\alpha0}\,\epsilon\,\eta0\,\xi0\right) \end{pmatrix}$$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$-2\,\text{Log}\left[e^{\frac{\alpha0\,(1+\epsilon)}{\epsilon}}\right]+4\,\text{Log}[1+\epsilon\,\eta1\,\xi1]==0.$$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$-2\,\text{Log}\left[e^{\frac{\alpha0\,(-1+\epsilon)}{\epsilon}}\right]-2\,\epsilon\,\text{Log}\left[e^{\frac{\alpha0\,(-1+\epsilon)}{\epsilon}}\right]-4\,\text{Log}[1+\epsilon\,\eta1\,\xi1]+4\,\epsilon\,\text{Log}[1+\epsilon\,\eta1\,\xi1]==0.$$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$-2\,\text{Log}\left[e^{\frac{\alpha0\,(-1+\epsilon)}{\epsilon}}\right]-\frac{2\,\text{Log}\left[e^{\frac{\alpha0\,(-1+\epsilon)}{\epsilon}}\right]}{\epsilon}+4\,\text{Log}[1+\epsilon\,\eta1\,\xi1]-\frac{4\,\text{Log}[1+\epsilon\,\eta1\,\xi1]}{\epsilon}==0.$$

General: Further output of Solve::incnst will be suppressed during this calculation.

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\eta0\to-\frac{\xi1}{1+\epsilon\,\eta1\,\xi1},\ \alpha0\to\frac{2\,\epsilon\,\text{Log}[1+\epsilon\,\eta1\,\xi1]}{1+\epsilon},\ \xi0\to-\frac{\eta1}{1+\epsilon\,\eta1\,\xi1}\right\}$$

```
FullSimplify[eqn1 /. sol]
```

$$\left\{\text{True}, \text{True}, \text{True}, \frac{-1+(1+\epsilon\,\eta1\,\xi1)^{\frac{2}{1+\epsilon}}}{1+\epsilon\,\eta1\,\xi1}==0\right\}$$