Pensieve header: The k-Logos in the quantized case; evolved from pensieve://Talks/Sydney-1708/nb/ExtraDetails.pdf.

Some Shortcuts

```
ME[x_] := MatrixExp[x]; MF[x_] := MatrixForm[x];
```

Representing ϵyax

(Borrowed from pensieve://2017-08/Multi-beta-yax.nb)

```
q = ⅇ^(ℏ γ ϵ); ρI = (1 0; 0 1); ρy = (0 0; -1 0); ρa = (0 0; 0 -γ);
ρx = (0 (-1+ⅇ^(-γ ϵ ℏ))/ℏ; 0 0); ρb = (ϵ 0; 0 0);
ρt = Simplify[ϵ ρa - γ ρb]; ρA = ME[-ℏ ϵ ρa]; ρB = ME[-ℏ γ ρb]; ρT = ME[ℏ ρt];
(# → MF@Simplify@ToExpression@#) & /@
 {"{{q}}", "ρy", "ρa", "ρx", "ρb", "ρt", "ρA", "ρB", "ρT", "ρI"}
```

$\{\{\{q\}\} \to \begin{pmatrix} e^{\gamma \epsilon \hbar} \end{pmatrix}, \rho y \to \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \rho a \to \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix}, \rho x \to \begin{pmatrix} 0 & \frac{-1+e^{-\gamma\epsilon\hbar}}{\hbar} \\ 0 & 0 \end{pmatrix}, \rho b \to \begin{pmatrix} \epsilon & 0 \\ 0 & 0 \end{pmatrix},$

$\rho t \to \begin{pmatrix} -\gamma\epsilon & 0 \\ 0 & -\gamma\epsilon \end{pmatrix}, \rho A \to \begin{pmatrix} 1 & 0 \\ 0 & e^{\gamma\epsilon\hbar} \end{pmatrix}, \rho B \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & 1 \end{pmatrix}, \rho T \to \begin{pmatrix} e^{-\gamma\epsilon\hbar} & 0 \\ 0 & e^{-\gamma\epsilon\hbar} \end{pmatrix}, \rho I \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$

```
{ρa.ρx - ρx.ρa == γ ρx, ρx.ρA == q ρA.ρx, ρa.ρy - ρy.ρa == -γ ρy,
  ρb.ρy - ρy.ρb == -ϵ ρy, ρx.ρy - q ρy.ρx == (ρI - ρT.ρA.ρA) / ℏ} // Simplify
```

{True, True, True, True, True}

Likely, the PBW property for ϵyax follows from the local injectivity below; meaning, that from the matrix below one may recover (τ, η, α, ξ).

```
Normal@Series[ME[ℏ τ ρt].ME[ℏ η ρy].ME[ℏ α ρa].ME[ℏ ξ ρx], {ℏ, 0, 1}] // MF
```

$\begin{pmatrix} 1 - \gamma\epsilon\tau\hbar & -\gamma\epsilon\xi\hbar \\ -\eta\hbar & 1 + (-\alpha\gamma - \gamma\epsilon\tau)\hbar \end{pmatrix}$

Some ϵyax lemmas

Lemma 1. $\mathbb{O}\left(e^{\alpha a+\xi x} \mid x\, a\right) = \mathbb{O}\left(e^{\alpha a+e^{-\gamma\alpha}\xi x} \mid a\, x\right)$.
Lemma 2. $\mathbb{O}(e^{\alpha a+\eta y} \mid a\, y) = \mathbb{O}(e^{\alpha a+e^{\gamma\alpha}\eta y} \mid y\, a)$.

Proofs.

```
MF /@ {lhs = ME[ξ ρx].ME[α ρa], rhs = ME[α ρa].ME[ⅇ^(-γ α) ξ ρx], lhs == rhs}
```

$\left\{\begin{pmatrix} 1 & -\frac{e^{-\alpha\gamma-\gamma\epsilon\hbar}\left(-1+e^{\gamma\epsilon\hbar}\right)\xi}{\hbar} \\ 0 & e^{-\alpha\gamma} \end{pmatrix}, \begin{pmatrix} 1 & -\frac{e^{-\alpha\gamma-\gamma\epsilon\hbar}\left(-1+e^{\gamma\epsilon\hbar}\right)\xi}{\hbar} \\ 0 & e^{-\alpha\gamma} \end{pmatrix}, \text{True}\right\}$

```
MF /@ {lhs = ME[α ρa].ME[η ρy], rhs = ME[e^(-γ α) η ρy].ME[α ρa], lhs == rhs}
```

$$\left\{\begin{pmatrix}1 & 0\\ -e^{-\alpha\gamma}\eta & e^{-\alpha\gamma}\end{pmatrix}, \begin{pmatrix}1 & 0\\ -e^{-\alpha\gamma}\eta & e^{-\alpha\gamma}\end{pmatrix}, \text{True}\right\}$$

Lemma 3 at $\lambda=0$. $\mathbb{O}(e^{\xi x+\eta y}\mid xy) = \mathbb{O}(\omega_0 e^{\eta_0 y+\alpha_0 a+\xi_0 x}\mid yax) = \mathbb{O}(e^{\tau_0 t+\eta_0 y+\alpha_0 a+\xi_0 x}\mid tyax)$, with

$$\left\{\omega_0\to\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar},\ \eta_0\to\frac{e^{\gamma\epsilon\hbar}\eta\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)},\ \alpha_0\to-\frac{2(\gamma\epsilon\hbar+\text{Log}[\hbar]-\text{Log}[-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)])}{\gamma},\ \xi_0\to\frac{e^{\gamma\epsilon\hbar}\xi\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)},\ \tau_0\to-\frac{\text{Log}\left[\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar}\right]}{\gamma\epsilon}\right\}$$

.

Derivation.

```
Column[
 sol =
  First@Solve[Thread[Flatten /@ (ME[ξ ρx].ME[η ρy] == ω0 ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx])],
     {ω0, η0, α0, ξ0}] /. (v_ → ℰ_) :> (v → FullSimplify@PowerExpand[ℰ]);
 AppendTo[sol, τ0 → Simplify[Log[ω0 /. sol]/(-γ ϵ)]]
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\omega0\to\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar}$$
$$\eta0\to\frac{e^{\gamma\epsilon\hbar}\eta\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)}$$
$$\alpha0\to-\frac{2(\gamma\epsilon\hbar+\text{Log}[\hbar]-\text{Log}[-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)])}{\gamma}$$
$$\xi0\to\frac{e^{\gamma\epsilon\hbar}\xi\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)}$$
$$\tau0\to-\frac{\text{Log}\left[\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar}\right]}{\gamma\epsilon}$$

Lemma 3 at $\epsilon=0$. With $\nu=(1+\delta t)^{-1}$, we have $\mathbb{O}(e^{\xi x+\eta y}\mid xy)=\mathbb{O}(\nu e^{\nu(\delta xy-\xi\eta t+\xi x+\eta y)}\mid tyx)$.

Derivation.

```
MapAt[Limit[#, ϵ → 0] &, sol, {All, 2}]
```

$\{\omega0\to 1,\ \eta0\to\eta,\ \alpha0\to 0,\ \xi0\to\xi,\ \tau0\to-\eta\xi\}$

And so at $\epsilon=0$, $\mathbb{O}(e^{\xi x+\eta y}\mid xy)=\mathbb{O}(e^{\xi x+\eta y-\xi\eta t}\mid tyx)$. Hence $\mathbb{O}(e^{\xi x+\eta y+\delta xy}\mid xy) = e^{\delta\partial_\xi\partial_\eta}\,\mathbb{O}(e^{\xi x+\eta y}\mid xy) = e^{\delta\partial_\xi\partial_\eta}\,\mathbb{O}(e^{\xi x+\eta y-\xi\eta t}\mid tyx) = \mathbb{O}(\psi\mid tyx)$, where $\psi = e^{\delta\partial_\xi\partial_\eta}e^{\xi x+\eta y-\xi\eta t}$ satisfies and is determined by $\psi_{\delta=0}=e^{\xi x+\eta y-\xi\eta t}$ and $\partial_\delta\psi=\partial_{\xi,\eta}\psi$.

```
With[{ψ = ν e^(ν (δ x y - ξ η t + ξ x + η y)) /. ν → (1 + δ t)^-1}, Simplify@{∂_δ ψ - ∂_{ξ,η} ψ, ψ /. δ → 0}]
```

$\{0,\ e^{y\eta+x\xi-t\eta\xi}\}$

Aside.

```
MapAt[Series[#, {ϵ, 0, 2}] &, sol, {All, 2}]
```

$$\left\{\omega 0\to 1+\gamma\eta\xi\epsilon-\frac{1}{2}\left(\gamma^2\eta\xi\hbar\right)\epsilon^2+O[\epsilon]^3,\ \eta 0\to\eta-\gamma\eta^2\xi\epsilon+\left(\gamma^2\eta^3\xi^2+\frac{1}{2}\gamma^2\eta^2\xi\hbar\right)\epsilon^2+O[\epsilon]^3,\right.$$
$$\alpha 0\to 2\eta\xi\epsilon+\left(-\gamma\eta^2\xi^2-\gamma\eta\xi\hbar\right)\epsilon^2+O[\epsilon]^3,\ \xi 0\to\xi-\gamma\eta\xi^2\epsilon+\left(\gamma^2\eta^2\xi^3+\frac{1}{2}\gamma^2\eta\xi^2\hbar\right)\epsilon^2+O[\epsilon]^3,$$
$$\left.\tau 0\to -\eta\xi+\frac{1}{2}\left(\gamma\eta^2\xi^2+\gamma\eta\xi\hbar\right)\epsilon+\frac{1}{6}\left(-2\gamma^2\eta^3\xi^3-3\gamma^2\eta^2\xi^2\hbar-\gamma^2\eta\xi\hbar^2\right)\epsilon^2+O[\epsilon]^3\right\}$$

```
MapAt[Simplify[# /. ħ → 1] &, sol, {All, 2}] // Column
```

$$\omega 0\to 1+\eta\left(\xi-e^{-\gamma\epsilon}\xi\right)$$
$$\eta 0\to\frac{e^{\gamma\epsilon}\eta}{-\eta\xi+e^{\gamma\epsilon}(1+\eta\xi)}$$
$$\alpha 0\to -2\epsilon+\frac{2\,\mathrm{Log}\left[-\eta\xi+e^{\gamma\epsilon}(1+\eta\xi)\right]}{\gamma}$$
$$\xi 0\to\frac{e^{\gamma\epsilon}\xi}{-\eta\xi+e^{\gamma\epsilon}(1+\eta\xi)}$$
$$\tau 0\to -\frac{\mathrm{Log}\left[1+\eta\left(\xi-e^{-\gamma\epsilon}\xi\right)\right]}{\gamma\epsilon}$$

A Lemma 3 for $yax_k := \epsilon yax/(\epsilon^{k+1}=0)$.

Lemma 3$_k$. $\mathbb{O}\left(e^{\xi x+\eta y+\delta xy}\mid xy\right)=\mathbb{O}\left(\nu e^{\nu(\delta xy-\xi\eta t+\xi x+\eta y)}\Lambda_k(\epsilon,y,a,x,\eta,\xi,\delta)\mid yax\right)$, with $\nu=(1+t\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon,y,a,x,\eta,\xi,\delta)$ is a fixed polynomial of degree at most $4k$ in y, $\sqrt{a}$, x, η, ξ, δ, with scalar coefficients.

Comment. And hence the yax_k invariant is computable in polynomial time.

Proof of Lemma 3$_k$. We know that $\mathbb{O}\left(e^{\xi x+\eta y}\mid xy\right)=\mathbb{O}\left(e^{\tau_0 t+\eta_0 y+\alpha_0 a+\xi_0 x}\mid yax\right)$, with

$$\left\{\eta_0\to\frac{e^{\gamma\epsilon\hbar}\eta\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)},\ \alpha_0\to-\frac{2\left(\gamma\epsilon\hbar+\mathrm{Log}[\hbar]-\mathrm{Log}\left[-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)\right]\right)}{\gamma},\ \xi_0\to\frac{e^{\gamma\epsilon\hbar}\xi\hbar}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)},\ \tau_0\to-\frac{\mathrm{Log}\left[\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar}\right]}{\gamma\epsilon}\right\}.$$

Expanding in ϵ we get $\mathbb{O}\left(e^{\xi x+\eta y}\mid xy\right)=\mathbb{O}\left(\lambda_\epsilon(\eta,\xi)\,\mathbb{O}\left(e^{\xi x+\eta y-\xi\eta t}\mid yax\right)\right)=\mathbb{O}\left(\lambda_\epsilon(\partial_y,\partial_x)\,e^{\xi x+\eta y-\xi\eta t}\mid yax\right)$ and so

$$\mathbb{O}\left(e^{\xi x+\eta y+\delta xy}\mid xy\right)=\mathbb{O}\left(\lambda_\epsilon(\partial_y,\partial_x)\,e^{\delta\partial_\xi\partial_\eta}e^{\xi x+\eta y-\xi\eta t}\mid yax\right)=\mathbb{O}\left(\nu\,\lambda_\epsilon(\partial_y,\partial_x)\,e^{\nu(\delta xy-\xi\eta t+\xi x+\eta y)}\mid yax\right).$$

```
DP_{η_→D_{y_}, ξ_→D_{x_}}[P_][λ_] :=
 Total[CoefficientRules[P, {η, ξ}] /. ({m_, n_} → c_) :> c D[λ, {y, m}, {x, n}]]
```

```
FullSimplify[e^(-ξ x-η y+ξ η t+τ0 t+η0 y+α0 a+ξ0 x) /. sol]
```

$$e^{-2a\epsilon\hbar+\frac{\eta\xi\left(y\eta+x\xi-t\eta\xi+e^{\gamma\epsilon\hbar}(-y\eta-x\xi+t\eta\xi+t\hbar)\right)}{-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)}}\hbar^{-\frac{2a}{\gamma}}\left(\frac{\eta\xi-e^{-\gamma\epsilon\hbar}\eta\xi+\hbar}{\hbar}\right)^{-\frac{t}{\gamma\epsilon}}\left(-\eta\xi+e^{\gamma\epsilon\hbar}(\eta\xi+\hbar)\right)^{\frac{2a}{\gamma}}$$

```
Normal@Series[e^(-ξ x-η y+ξ η t+τ0 t+η0 y+α0 a+ξ0 x) /. sol, {ϵ, 0, 2}]
```

$$1+\frac{1}{2}\epsilon\left(4a\eta\xi-2y\gamma\eta^2\xi-2x\gamma\eta\xi^2+t\gamma\eta^2\xi^2+t\gamma\eta\xi\hbar\right)+$$
$$\frac{1}{2}\epsilon^2\left(\frac{1}{4}\left(4a\eta\xi-2y\gamma\eta^2\xi-2x\gamma\eta\xi^2+t\gamma\eta^2\xi^2+t\gamma\eta\xi\hbar\right)^2+\frac{1}{3}\left(-6a\gamma\eta^2\xi^2+6y\gamma^2\eta^3\xi^2+\right.\right.$$
$$\left.\left.6x\gamma^2\eta^2\xi^3-2t\gamma^2\eta^3\xi^3-6a\gamma\eta\xi\hbar+3y\gamma^2\eta^2\xi\hbar+3x\gamma^2\eta\xi^2\hbar-3t\gamma^2\eta^2\xi^2\hbar-t\gamma^2\eta\xi\hbar^2\right)\right)$$

$$\textbf{Expand}\Big[e^{-\nu\,(\delta\,x\,y-\xi\,\eta\,t+\xi\,x+\eta\,y)}\ \textbf{DP}_{\eta\to D_y,\,\xi\to D_x}\Big[\textbf{Normal@Series}\Big[e^{-\xi\,x-\eta\,y+\xi\,\eta\,t+\tau 0\,t+\eta 0\,y+\alpha 0\,a+\xi 0\,x}\ /.\ \textbf{sol},\ \{\epsilon,0,1\}\Big]\Big]\Big[e^{\nu\,(\delta\,x\,y-\xi\,\eta\,t+\xi\,x+\eta\,y)}\Big]\Big]$$

$$1+2\,a\,\delta\,\epsilon\,\nu+2\,a\,x\,y\,\delta^2\,\epsilon\,\nu^2+t\,\gamma\,\delta^2\,\epsilon\,\nu^2-4\,x\,y\,\gamma\,\delta^2\,\epsilon\,\nu^2+2\,a\,y\,\delta\,\epsilon\,\eta\,\nu^2-2\,y\,\gamma\,\delta\,\epsilon\,\eta\,\nu^2+2\,t\,x\,y\,\gamma\,\delta^3\,\epsilon\,\nu^3-2\,x^2\,y^2\,\gamma\,\delta^3\,\epsilon\,\nu^3+2\,t\,y\,\gamma\,\delta^2\,\epsilon\,\eta\,\nu^3-3\,x\,y^2\,\gamma\,\delta^2\,\epsilon\,\eta\,\nu^3-y^2\,\gamma\,\delta\,\epsilon\,\eta^2\,\nu^3+\frac{1}{2}\,t\,x^2\,y^2\,\gamma\,\delta^4\,\epsilon\,\nu^4+t\,x\,y^2\,\gamma\,\delta^3\,\epsilon\,\eta\,\nu^4+\frac{1}{2}\,t\,y^2\,\gamma\,\delta^2\,\epsilon\,\eta^2\,\nu^4+2\,a\,x\,\delta\,\epsilon\,\nu^2\,\xi-2\,x\,\gamma\,\delta\,\epsilon\,\nu^2\,\xi+2\,a\,\epsilon\,\eta\,\nu^2\,\xi+2\,t\,x\,\gamma\,\delta^2\,\epsilon\,\nu^3\,\xi-3\,x^2\,y\,\gamma\,\delta^2\,\epsilon\,\nu^3\,\xi+2\,t\,\gamma\,\delta\,\epsilon\,\eta\,\nu^3\,\xi-4\,x\,y\,\gamma\,\delta\,\epsilon\,\eta\,\nu^3\,\xi-y\,\gamma\,\epsilon\,\eta^2\,\nu^3\,\xi+t\,x^2\,y\,\gamma\,\delta^3\,\epsilon\,\nu^4\,\xi+2\,t\,x\,y\,\gamma\,\delta^2\,\epsilon\,\eta\,\nu^4\,\xi+t\,y\,\gamma\,\delta\,\epsilon\,\eta^2\,\nu^4\,\xi-x^2\,\gamma\,\delta\,\epsilon\,\nu^3\,\xi^2-x\,\gamma\,\epsilon\,\eta\,\nu^3\,\xi^2+\frac{1}{2}\,t\,x^2\,\gamma\,\delta^2\,\epsilon\,\nu^4\,\xi^2+t\,x\,\gamma\,\delta\,\epsilon\,\eta\,\nu^4\,\xi^2+\frac{1}{2}\,t\,\gamma\,\epsilon\,\eta^2\,\nu^4\,\xi^2+\frac{1}{2}\,t\,\gamma\,\delta\,\epsilon\,\nu\,\hbar+\frac{1}{2}\,t\,x\,y\,\gamma\,\delta^2\,\epsilon\,\nu^2\,\hbar+\frac{1}{2}\,t\,y\,\gamma\,\delta\,\epsilon\,\eta\,\nu^2\,\hbar+\frac{1}{2}\,t\,x\,\gamma\,\delta\,\epsilon\,\nu^2\,\xi\,\hbar+\frac{1}{2}\,t\,\gamma\,\epsilon\,\eta\,\nu^2\,\xi\,\hbar$$

$$\Lambda_{k\_}:=\textbf{Collect}\Big[e^{-\nu\,(\delta\,x\,y-\xi\,\eta\,t+\xi\,x+\eta\,y)}\ \textbf{DP}_{\eta\to D_y,\,\xi\to D_x}\Big[\textbf{Normal@Series}\Big[e^{-\xi\,x-\eta\,y+\xi\,\eta\,t+\tau 0\,t+\eta 0\,y+\alpha 0\,a+\xi 0\,x}\ /.\ \textbf{sol},\ \{\epsilon,0,k\}\Big]\Big]\Big[e^{\nu\,(\delta\,x\,y-\xi\,\eta\,t+\xi\,x+\eta\,y)}\Big],\ \epsilon,\ \textbf{Simplify}\Big];$$

$$\Lambda_0$$

$$1$$

$$\Lambda_1$$

$$1+\frac{1}{2}\,\epsilon\,\nu\,\Big(4\,a\,\big(\delta+x\,y\,\delta^2\,\nu+y\,\delta\,\eta\,\nu+x\,\delta\,\nu\,\xi+\eta\,\nu\,\xi\big)-2\,\gamma\,\nu\,\big(y\,\eta+x\,(2\,y\,\delta+\xi)\big)\,\big(x\,y\,\delta^2\,\nu+\eta\,\nu\,\xi+\delta\,(2+y\,\eta\,\nu+x\,\nu\,\xi)\big)+t\,\gamma\,\Big(x^2\,y^2\,\delta^4\,\nu^3+2\,x\,y\,\delta^3\,\nu^2\,(2+y\,\eta\,\nu+x\,\nu\,\xi)+\eta\,\nu\,\xi\,(\eta\,\nu^2\,\xi+\hbar)+\delta^2\,\nu\,\big(2+y^2\,\eta^2\,\nu^2+4\,x\,\nu\,\xi+x^2\,\nu^2\,\xi^2+4\,y\,\eta\,\nu\,(1+x\,\nu\,\xi)+x\,y\,\hbar\big)+\delta\,\big(2\,y\,\eta^2\,\nu^3\,\xi+\hbar+x\,\nu\,\xi\,\hbar+\eta\,\nu\,(4\,\nu\,\xi+2\,x\,\nu^2\,\xi^2+y\,\hbar)\big)\Big)\Big)$$

Λ_2

$$1+\frac{1}{24}\epsilon^2\nu\,(48\,a^2\nu\,(2\delta^2+4\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)-$$
$$24\,a\gamma\nu\,(2\delta^2+4\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)-$$
$$48\,a\,y\gamma\,(x\delta+\eta)\,\nu^2\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)+$$
$$24\,y\gamma^2\,(x\delta+\eta)\,\nu^2\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)-$$
$$48\,a\,x\gamma\nu^2\,(y\delta+\xi)\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)+$$
$$24\,x\gamma^2\nu^2\,(y\delta+\xi)\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)+$$
$$12\,y^2\gamma^2\,(x\delta+\eta)^2\nu^3\,(12\delta^2+8\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)+$$
$$12\,x^2\gamma^2\nu^3\,(y\delta+\xi)^2\,(12\delta^2+8\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)+24\,a\,t\gamma\nu^2$$
$$(6\delta^3+18\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+9\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)-8\,t\gamma^2$$
$$\nu^2\,(6\delta^3+18\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+9\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)+24\,x$$
$$y\gamma^2\nu^2\,(6\delta^3+18\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+9\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)-$$
$$12\,t\,y\gamma^2\,(x\delta+\eta)\,\nu^3\,(24\delta^3+36\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+$$
$$12\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)-12\,t\,x\gamma^2\nu^3\,(y\delta+\xi)$$
$$(24\delta^3+36\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+12\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)+$$
$$3\,t^2\gamma^2\nu^3\,(24\delta^4+96\delta^3\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+72\delta^2\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+16\delta\,(x\delta+\eta)^3$$
$$\nu^3\,(y\delta+\xi)^3+(x\delta+\eta)^4\nu^4\,(y\delta+\xi)^4)-24\,a\gamma\,(\delta+x\,y\delta^2\nu+y\delta\eta\nu+x\delta\nu\xi+\eta\nu\xi)\,\hbar+$$
$$24\,a\,t\gamma\nu\,(2\delta^2+4\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)\,\hbar-$$
$$12\,t\gamma^2\nu\,(2\delta^2+4\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)\,\hbar-$$
$$12\,t\,y\gamma^2\,(x\delta+\eta)\,\nu^2\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)\,\hbar-$$
$$12\,t\,x\gamma^2\nu^2\,(y\delta+\xi)\,(6\delta^2+6\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)\,\hbar+$$
$$6\,t^2\gamma^2\nu^2\,(6\delta^3+18\delta^2\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+9\delta\,(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2+(x\delta+\eta)^3\nu^3\,(y\delta+\xi)^3)$$
$$\hbar+12\,y\gamma^2\,(x\delta+\eta)\,\nu\,(x\,y\delta^2\nu+\eta\nu\xi+\delta\,(2+y\eta\nu+x\nu\xi))\,\hbar+$$
$$12\,x\gamma^2\nu\,(y\delta+\xi)\,(x\,y\delta^2\nu+\eta\nu\xi+\delta\,(2+y\eta\nu+x\nu\xi))\,\hbar-$$
$$4\,t\gamma^2\,(\delta+x\,y\delta^2\nu+y\delta\eta\nu+x\delta\nu\xi+\eta\nu\xi)\,\hbar^2+$$
$$3\,t^2\gamma^2\nu\,(2\delta^2+4\delta\,(x\delta+\eta)\,\nu\,(y\delta+\xi)+(x\delta+\eta)^2\nu^2\,(y\delta+\xi)^2)\,\hbar^2)+$$
$$\frac{1}{2}\epsilon\nu\,(4\,a\,(\delta+x\,y\delta^2\nu+y\delta\eta\nu+x\delta\nu\xi+\eta\nu\xi)-2\gamma\nu\,(y\eta+x\,(2\,y\delta+\xi))$$
$$(x\,y\delta^2\nu+\eta\nu\xi+\delta\,(2+y\eta\nu+x\nu\xi))+t\gamma\,(x^2y^2\delta^4\nu^3+2\,x\,y\delta^3\nu^2\,(2+y\eta\nu+x\nu\xi)+$$
$$\eta\nu\xi\,(\eta\nu^2\xi+\hbar)+\delta^2\nu\,(2+y^2\eta^2\nu^2+4\,x\nu\xi+x^2\nu^2\xi^2+4\,y\eta\nu\,(1+x\nu\xi)+x\,y\hbar)+$$
$$\delta\,(2\,y\eta^2\nu^3\xi+\hbar+x\nu\xi\hbar+\eta\nu\,(4\nu\xi+2\,x\nu^2\xi^2+y\hbar)))))$$

```
Collect[Λ2 /. ν → (1 + δ t)^-1, ϵ, Simplify]
```

$$1+\frac{1}{24\,(1+t\delta)^8}$$
$$\epsilon^2\,(48\,a^2\,(1+t\delta)^4\,(2\delta^2\,(1+t\delta)^2+4\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+(x\delta+\eta)^2\,(y\delta+\xi)^2)-$$
$$24\,a\gamma\,(1+t\delta)^4\,(2\delta^2\,(1+t\delta)^2+4\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+(x\delta+\eta)^2\,(y\delta+\xi)^2)-$$
$$48\,a\,y\gamma\,(1+t\delta)^3\,(x\delta+\eta)$$
$$(6\delta^2\,(1+t\delta)^2+6\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+(x\delta+\eta)^2\,(y\delta+\xi)^2)+24\,y\gamma^2\,(1+t\delta)^3$$
$$(x\delta+\eta)\,(6\delta^2\,(1+t\delta)^2+6\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+(x\delta+\eta)^2\,(y\delta+\xi)^2)-48\,a\,x\gamma$$
$$(1+t\delta)^3\,(y\delta+\xi)\,(6\delta^2\,(1+t\delta)^2+6\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+(x\delta+\eta)^2\,(y\delta+\xi)^2)+$$
$$24\,x\gamma^2\,(1+t\delta)^3\,(y\delta+\xi)\,(6\delta^2\,(1+t\delta)^2+6\delta\,(1+t\delta)\,(x\delta+\eta)\,(y\delta+\xi)+$$
$$(x\delta+\eta)^2\,(y\delta+\xi)^2)+12\,y^2\gamma^2\,(1+t\delta)^2\,(x\delta+\eta)^2$$

$$\left(12\,\delta^2\,(1+t\,\delta)^2+8\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+12\,x^2\,\gamma^2$$
$$(1+t\,\delta)^2\,(y\,\delta+\xi)^2\,\left(12\,\delta^2\,(1+t\,\delta)^2+8\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)+$$
$$24\,a\,t\,\gamma\,(1+t\,\delta)^2\,\Big(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)-$$
$$8\,t\,(\gamma+t\,\gamma\,\delta)^2\,\Big(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)+$$
$$24\,x\,y\,(\gamma+t\,\gamma\,\delta)^2\,\Big(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)-$$
$$12\,t\,y\,\gamma^2\,(1+t\,\delta)\,(x\,\delta+\eta)\,\Big(24\,\delta^3\,(1+t\,\delta)^3+36\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$12\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)-$$
$$12\,t\,x\,\gamma^2\,(1+t\,\delta)\,(y\,\delta+\xi)\,\Big(24\,\delta^3\,(1+t\,\delta)^3+36\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$12\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)+$$
$$3\,t^2\,\gamma^2\,\Big(24\,\delta^4\,(1+t\,\delta)^4+96\,\delta^3\,(1+t\,\delta)^3\,(x\,\delta+\eta)\,(y\,\delta+\xi)+72\,\delta^2\,(1+t\,\delta)^2$$
$$(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+16\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3+(x\,\delta+\eta)^4\,(y\,\delta+\xi)^4\Big)+$$
$$24\,a\,t\,\gamma\,(1+t\,\delta)^4\,\left(2\,\delta^2\,(1+t\,\delta)^2+4\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)\hbar-$$
$$12\,t\,\gamma^2\,(1+t\,\delta)^4\,\left(2\,\delta^2\,(1+t\,\delta)^2+4\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)\hbar-$$
$$12\,t\,y\,\gamma^2\,(1+t\,\delta)^3\,(x\,\delta+\eta)$$
$$\left(6\,\delta^2\,(1+t\,\delta)^2+6\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)\hbar-12\,t\,x\,\gamma^2\,(1+t\,\delta)^3$$
$$(y\,\delta+\xi)\,\left(6\,\delta^2\,(1+t\,\delta)^2+6\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)\hbar+$$
$$6\,t^2\,\gamma^2\,(1+t\,\delta)^2\,\Big(6\,\delta^3\,(1+t\,\delta)^3+18\,\delta^2\,(1+t\,\delta)^2\,(x\,\delta+\eta)\,(y\,\delta+\xi)+$$
$$9\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2+(x\,\delta+\eta)^3\,(y\,\delta+\xi)^3\Big)\hbar-$$
$$24\,a\,\gamma\,(1+t\,\delta)^6\,\left((t+x\,y)\,\delta^2+\eta\,\xi+\delta\,(1+y\,\eta+x\,\xi)\right)\hbar+$$
$$12\,y\,\gamma^2\,(1+t\,\delta)^5\,(x\,\delta+\eta)\,\left(x\,y\,\delta^2+\eta\,\xi+\delta\,(2+2\,t\,\delta+y\,\eta+x\,\xi)\right)\hbar+$$
$$12\,x\,\gamma^2\,(1+t\,\delta)^5\,(y\,\delta+\xi)\,\left(x\,y\,\delta^2+\eta\,\xi+\delta\,(2+2\,t\,\delta+y\,\eta+x\,\xi)\right)\hbar+$$
$$3\,t^2\,\gamma^2\,(1+t\,\delta)^4\,\left(2\,\delta^2\,(1+t\,\delta)^2+4\,\delta\,(1+t\,\delta)\,(x\,\delta+\eta)\,(y\,\delta+\xi)+(x\,\delta+\eta)^2\,(y\,\delta+\xi)^2\right)\hbar^2-$$
$$4\,t\,\gamma^2\,(1+t\,\delta)^6\,\left((t+x\,y)\,\delta^2+\eta\,\xi+\delta\,(1+y\,\eta+x\,\xi)\right)\hbar^2\Big)+$$
$$\frac{1}{2\,(1+t\,\delta)^4}\,\epsilon\,\Big(4\,a\,(1+t\,\delta)^2\,\left((t+x\,y)\,\delta^2+\eta\,\xi+\delta\,(1+y\,\eta+x\,\xi)\right)+$$
$$\gamma\,\Big(-2\,\big(y\,\eta\,(\delta\,(2+y\,\eta)+\eta\,\xi)+x^2\,\delta\,(2\,y^2\,\delta^2+3\,y\,\delta\,\xi+\xi^2)+$$
$$x\,(3\,y^2\,\delta^2\,\eta+4\,y\,\delta\,(\delta+\eta\,\xi)+\xi\,(2\,\delta+\eta\,\xi))\big)+t^4\,\delta^4\,\hbar+t^3\,\delta^2\,\big(\eta\,\xi\,\hbar+\delta\,(3+y\,\eta+x\,\xi)\,\hbar+$$
$$\delta^2\,(2+x\,y\,\hbar)\big)+t\,\big(-3\,x^2\,y^2\,\delta^4-4\,x\,y\,\delta^3\,(3+y\,\eta+x\,\xi)+\eta\,\xi\,(\eta\,\xi+\hbar)-$$
$$\delta^2\,\left(-2+y^2\,\eta^2+4\,x\,\xi+x^2\,\xi^2+4\,y\,\eta\,(1+x\,\xi)-x\,y\,\hbar\right)+\delta\,(4\,\eta\,\xi+\hbar+y\,\eta\,\hbar+x\,\xi\,\hbar)\big)+$$
$$t^2\,\delta\,\left(-4\,x\,y\,\delta^3+2\,\eta\,\xi\,\hbar+2\,\delta^2\,(2+x\,y\,\hbar)+\delta\,(4\,\eta\,\xi+3\,\hbar+2\,y\,\eta\,\hbar+2\,x\,\xi\,\hbar)\right)\Big)\Big)$$

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2k+2$ in said variables.

The Main $\mathfrak{g}_k$ Theorem

The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T)=\mathbb{O}\left(\omega\,e^{L+Q+P}\;\middle|\;\prod_{i\in S} e_i\,l_i\,f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i=e^{h_i}$), where $L=\sum a_{ij}\,h_i\,l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q=\sum b_{ij}\,e_i\,f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2\,d+2$ in $\left\{e_i,\sqrt{l_i}\,,\ f_i\right\}$. Furthermore, after setting $h_i=h$ and $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^{\pm}=?$, $n^{\pm}=?$.

1. $\mathbb{O}\left(\mathcal{P}(l,\,e)\,e^{\gamma\,l+\beta\,e}\;\middle|\;l\,e\right)=\mathbb{O}\left(\mathcal{P}(\partial_\gamma,\,\partial_\beta)\,e^{\gamma\,l+e^{\gamma}\,\beta\,e}\;\middle|\;e\,l\right)$,

2. $\mathbb{O}\left(\mathcal{P}(l,\,f)\,e^{\gamma\,l+\beta\,f}\;\middle|\;f\,l\right)=\mathbb{O}\left(\mathcal{P}(\partial_\gamma,\,\partial_\beta)\,e^{\gamma\,l+e^{\gamma}\,\beta\,f}\;\middle|\;l\,f\right)$,

3. $\mathbb{O}\left(\mathcal{P}(e,\,f)\,e^{\beta\,e+\alpha\,f+\delta\,e\,f}\;\middle|\;f\,e\right)=\mathbb{O}\left(\nu\,\mathcal{P}(\partial_\beta,\,\partial_\alpha)\,e^{\nu(-\alpha\,\beta\,h+\beta\,e+\alpha\,f+\delta\,e\,f)}\,\Lambda_k(\epsilon,\,e,\,l,\,f,\,\alpha,\,\beta,\,\delta)\;\middle|\;e\,f\right)$, with $\nu=(1+\mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon,\,e,\,l,\,f,\,\alpha,\,\beta,\,\delta)$ as above.

Pushforwards of distributions, 0-dimensional QFT, Feynman diagrams and what had really happened here.