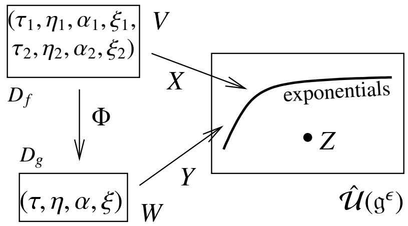
Cheat Sheet Pushforwards



Definition. PS :=(Power Series).

For a vector space V, let $\mathcal{D}_0(V)$ denote the space of distributions on V whose support is $\{0\}$. Via the Laplace transform $\mathcal{D}_0(V)$ can be identified with $\mathcal{S}(V)$; we have $\mathcal{L}_V\colon \mathcal{D}_0(V) \to \mathcal{S}(V)$.

Challenge. With $\Phi\colon V \to W$ a PS map near 0 (so $\Phi \in \mathrm{mor}_{PS}(V \to W) := W \otimes \mathcal{S}^+(V^*)$ and with $D_f \in \mathcal{D}_0(V)$, understand $\Phi_* D_f \in \mathcal{D}_0(W)$.

Challenge. With $\Phi = (\phi_j(\alpha_i))$ and $Z = \zeta(\partial_{\alpha_i})$, set $\Phi_* Z := \mathbb{e}^{\sum \partial_{\beta_j}\phi_j(\partial_{a_i})}\zeta(a_i)\big|_{a_i=0}$. With $(a_i, y_i, x_i, t_i) := (\partial_{\alpha_i}, \partial_{\eta_i}, \partial_{\xi_i}, \partial_{\tau_i})$, compute/implement $\Phi_* Z$, with

$$Z = \omega \exp\left(\sum \lambda_{ij} t_i a_j + \sum q_{ij} y_i x_j + \epsilon P_0\right),$$

$\lambda_{ij} \in \mathbb{Z}$, $\omega, q_{ij} \in R := \mathbb{Q}(T_i = \mathbb{e}^{t_i})$, $P_0 \in R[a_i, y_i, x_i]$, and

$$\begin{aligned}
\Phi^*(\bar\alpha_i) &= \sum \psi^1_{ij}\alpha_j + \epsilon P_1, \\
\Phi^*(\bar\eta_i) &= \sum \psi^2_{ij}\eta_j + \epsilon P_2, \\
\Phi^*(\bar\xi_i) &= \sum \psi^3_{ij}\xi_j + \epsilon P_3, \\
\Phi^*(\bar\tau_i) &= \sum \psi^4_{ij}\tau_j + \sum \gamma_{ij}\eta_i\xi_j + \epsilon P_4,
\end{aligned}$$

$\psi^{1,4}_{ij} \in \mathbb{Z}$, $\psi^{2,3} \in R$, $P_{1,4} \in \mathbb{Q}[x_i, y_i]$, $P_{2,3} \in R[x_i, y_i]$, $\gamma_{ij} \in R$.

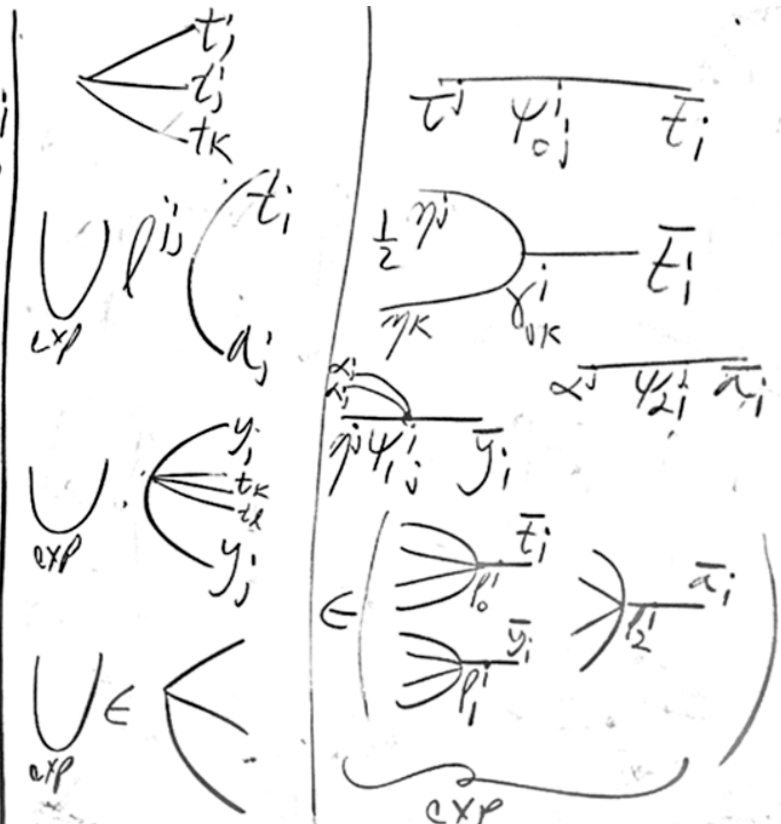
Example. 2017-07/Multi-beta-yax.nb: In $\mathcal{U}_{\gamma^{-1};\gamma\beta}$ where $q = \mathbb{e}^\beta$, $\prod_{i=1}^2 \mathbb{e}^{\eta_i y}\mathbb{e}^{\alpha_i a}\mathbb{e}^{\xi_i x} = \mathbb{e}^{\eta y}\mathbb{e}^{\alpha a}\mathbb{e}^{\xi x}\mathbb{e}^{\tau t}$, with

$$\begin{aligned}
\eta &= \eta_1 + \eta_2 e^{-\gamma\alpha_1} - \beta\gamma\eta_2^2\xi_1 e^{-\gamma\alpha_1} + \ldots = \eta_1 + \delta\eta_2 e^{\beta-\alpha_1\gamma} \\
\alpha &= \alpha_1 + \alpha_2 + 2\beta\eta_2\xi_1 + \ldots = \alpha_1 + \alpha_2 - 2(\beta + \log\delta)/\gamma \\
\xi &= \xi_1 e^{-\gamma\alpha_2} + \xi_2 - \beta\gamma\eta_2\xi_1^2 e^{-\gamma\alpha_2} + \ldots = \delta\xi_1 e^{\beta-\alpha_2\gamma} + \xi_2 \\
\tau &= -\eta_2\xi_1 + \beta\eta_2\xi_1(\gamma\eta_2\xi_1 + 1)/2 + \ldots = (\beta + \log\delta)/(\beta\gamma)
\end{aligned}$$

and $\delta := \left(\left(e^\beta - 1\right)\gamma\eta_2\xi_1 + e^\beta\right)^{-1} = 1 - (1 + \gamma\eta_1\xi_1)\beta + \ldots$.

Problem: Compute $\tilde\Phi_* \tilde Z$, where

$$\Phi^*(\bar\tau^i) = \psi^i_{0j}\tau^j + \tfrac12\gamma^i_{jk}\eta^j\eta^k + \epsilon P_0^i$$
$$\Phi^*(\bar\eta^i) = \psi^i_{1j}\eta^j + \epsilon P_1^i$$
$$\Phi^*(\bar\alpha^i) = \psi^i_{2j}\alpha^j + \epsilon P_2^i$$
$$\tilde Z = \omega e^{L + \frac12 Q + \epsilon P}$$
$$L = l^{jk} t_j a_k \qquad Q = q^{jk} y_j y_k$$



$$\Phi: \mathbb{R}^{n_0+n_1+n_2}_{\tau^j,\eta^j,\alpha^j} \longrightarrow \mathbb{R}^{m_0+m_1+m_2}_{\bar\tau^i,\bar\eta^i,\bar\alpha^i}$$