Pensieve header: Products of multiple exponentials in the β-yax algebra; continues pensieve://2017-06/BW.nb and pensieve://2017-07/.

```
ME = MatrixExp; MF = MatrixForm;
```

Representing β-yax

```
ℏ = (σ - ρ)/(γ β); q = e^(ℏ γ β);
y = {{0, 0}, {-e^ρ, 0}}; a = γ/(ρ - σ) {{ρ, 0}, {0, σ}}; x = (e^-σ - e^-ρ)/ℏ {{0, 1}, {0, 0}}; b = β/(σ - ρ) {{σ, 0}, {0, ρ}};
MF /@ {A = ME[-ℏ β a], B = ME[-ℏ γ b], t = Simplify[β a - γ b] , T = ME[ℏ t]}
```

$$\left\{\begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{\sigma}\end{pmatrix}, \begin{pmatrix} e^{-\sigma} & 0 \\ 0 & e^{-\rho}\end{pmatrix}, \begin{pmatrix} \frac{\beta\gamma(\rho+\sigma)}{\rho-\sigma} & 0 \\ 0 & \frac{\beta\gamma(\rho+\sigma)}{\rho-\sigma}\end{pmatrix}, \begin{pmatrix} e^{-\rho-\sigma} & 0 \\ 0 & e^{-\rho-\sigma}\end{pmatrix}\right\}$$

```
{a.x - x.a == γ x, x.A == q A.x, a.y - y.a == -γ y,
  b.y - y.b == -β y, x.y - q y.x == ({{1, 0}, {0, 1}} - T.A.A)/ℏ} // Simplify
```

{True, True, True, True, True}

```
Clear[ℏ]; ρ = 0; σ = ℏ β γ;
t = Simplify[β a - γ b]; A = ME[-β a / γ]; B = ME[-b]; T = ME[t / γ];
(# → MF@Simplify@ToExpression@#) & /@ {"{{q}}", "y", "a", "x", "b", "t", "A", "B", "T"}
```

$$\left\{\{\{q\}\} \to \begin{pmatrix} e^{\beta\gamma\hbar}\end{pmatrix}, y \to \begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix}, a \to \begin{pmatrix} 0 & 0 \\ 0 & -\gamma\end{pmatrix}, x \to \begin{pmatrix} 0 & \frac{-1+e^{-\beta\gamma\hbar}}{\hbar} \\ 0 & 0\end{pmatrix},\right.$$
$$\left. b \to \begin{pmatrix} \beta & 0 \\ 0 & 0\end{pmatrix}, t \to \begin{pmatrix} -\beta\gamma & 0 \\ 0 & -\beta\gamma\end{pmatrix}, A \to \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta}\end{pmatrix}, B \to \begin{pmatrix} e^{-\beta} & 0 \\ 0 & 1\end{pmatrix}, T \to \begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta}\end{pmatrix}\right\}$$

The 2-Stitch

```
P = {{1, 0}, {0, 1}};
Do[P = Expand[P.ME[η_i y].ME[α_i a].ME[ξ_i x]], {i, 2}];
Column[
 sol = First@Solve[Thread[Flatten /@ (P == τo ME[ηo y].ME[αo a].ME[ξo x])], {τo, ηo, αo, ξo}] /.
   (v_ → ℰ_) :> (v → Simplify@PowerExpand[ℰ])
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\tau o \to 1 + \frac{\left(1-e^{-\beta\gamma\hbar}\right)\eta_2\,\xi_1}{\hbar}$$
$$\eta o \to \eta_1 + \frac{e^{\beta\gamma\hbar-\gamma\alpha_1}\,\hbar\,\eta_2}{e^{\beta\gamma\hbar}\,\hbar+\left(-1+e^{\beta\gamma\hbar}\right)\eta_2\,\xi_1}$$
$$\alpha o \to -\frac{2\left(\beta\gamma\hbar+\mathrm{Log}[\hbar]-\mathrm{Log}\left[e^{\beta\gamma\hbar}\,\hbar+\left(-1+e^{\beta\gamma\hbar}\right)\eta_2\,\xi_1\right]\right)}{\gamma} + \alpha_1 + \alpha_2$$
$$\xi o \to \frac{e^{-\gamma\alpha_2}\left(e^{\gamma(\beta\hbar+\alpha_2)}\,\hbar\,\xi_2+\xi_1\left(e^{\beta\gamma\hbar}\,\hbar+e^{\gamma\alpha_2}\left(-1+e^{\beta\gamma\hbar}\right)\eta_2\,\xi_2\right)\right)}{e^{\beta\gamma\hbar}\,\hbar+\left(-1+e^{\beta\gamma\hbar}\right)\eta_2\,\xi_1}$$

```
Column@Series[{ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol, {β, 0, 2}]
```

$\left(\eta_1 + e^{-\gamma \alpha_1} \eta_2\right) - e^{-\gamma \alpha_1} \gamma \eta_2^2 \xi_1 \beta + \frac{1}{2} e^{-\gamma \alpha_1} \gamma^2 \eta_2^2 \xi_1 \left(\hbar + 2 \eta_2 \xi_1\right) \beta^2 + O[\beta]^3$

$(\alpha_1 + \alpha_2) + 2 \eta_2 \xi_1 \beta + \left(-\gamma \hbar \eta_2 \xi_1 - \gamma \eta_2^2 \xi_1^2\right) \beta^2 + O[\beta]^3$

$\left(e^{-\gamma \alpha_2} \xi_1 + \xi_2\right) - e^{-\gamma \alpha_2} \gamma \eta_2 \xi_1^2 \beta + \frac{1}{2} e^{-\gamma \alpha_2} \gamma^2 \eta_2 \xi_1^2 \left(\hbar + 2 \eta_2 \xi_1\right) \beta^2 + O[\beta]^3$

$1 + \gamma \eta_2 \xi_1 \beta - \frac{1}{2} \left(\gamma^2 \hbar \eta_2 \xi_1\right) \beta^2 + O[\beta]^3$

$-\eta_2 \xi_1 + \frac{1}{2} \left(\gamma \hbar \eta_2 \xi_1 + \gamma \eta_2^2 \xi_1^2\right) \beta + \frac{1}{6} \left(-\gamma^2 \hbar^2 \eta_2 \xi_1 - 3 \gamma^2 \hbar \eta_2^2 \xi_1^2 - 2 \gamma^2 \eta_2^3 \xi_1^3\right) \beta^2 + O[\beta]^3$

xy to yax

```
Column[
 sol1 =
  First@Solve[Thread[Flatten /@ (ME[ξ x].ME[η y] == τo ME[ηo y].ME[αo a].ME[ξo x])], {τo, ηo, αo, ξo}] /.
   (v_ → ℰ_) :> (v → FullSimplify@PowerExpand[ℰ])
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$\tau o \to \frac{\eta \xi - e^{-\beta \gamma \hbar} \eta \xi + \hbar}{\hbar}$

$\eta o \to \frac{e^{\beta \gamma \hbar} \eta \hbar}{-\eta \xi + e^{\beta \gamma \hbar} (\eta \xi + \hbar)}$

$\alpha o \to -\frac{2 \left(\beta \gamma \hbar + \mathrm{Log}[\hbar] - \mathrm{Log}\left[-\eta \xi + e^{\beta \gamma \hbar} (\eta \xi + \hbar)\right]\right)}{\gamma}$

$\xi o \to \frac{e^{\beta \gamma \hbar} \xi \hbar}{-\eta \xi + e^{\beta \gamma \hbar} (\eta \xi + \hbar)}$

```
Column@Series[{ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol1, {β, 0, 2}]
```

$\eta - \gamma \eta^2 \xi \beta + \left(\gamma^2 \eta^3 \xi^2 + \frac{1}{2} \gamma^2 \eta^2 \xi \hbar\right) \beta^2 + O[\beta]^3$

$2 \eta \xi \beta + \left(-\gamma \eta^2 \xi^2 - \gamma \eta \xi \hbar\right) \beta^2 + O[\beta]^3$

$\xi - \gamma \eta \xi^2 \beta + \left(\gamma^2 \eta^2 \xi^3 + \frac{1}{2} \gamma^2 \eta \xi^2 \hbar\right) \beta^2 + O[\beta]^3$

$1 + \gamma \eta \xi \beta - \frac{1}{2} \left(\gamma^2 \eta \xi \hbar\right) \beta^2 + O[\beta]^3$

$-\eta \xi + \frac{1}{2} \left(\gamma \eta^2 \xi^2 + \gamma \eta \xi \hbar\right) \beta + \frac{1}{6} \left(-2 \gamma^2 \eta^3 \xi^3 - 3 \gamma^2 \eta^2 \xi^2 \hbar - \gamma^2 \eta \xi \hbar^2\right) \beta^2 + O[\beta]^3$