Pensieve header: The bch co-product for the 2D Lie algebra.

```
𝔼[M_] /; MatrixQ[M] := MatrixExp[M];
a = (γ 0; 0 0); x = (0 1; 0 0);
a.x - x.a == γ x
```

True

```
𝔼[α a + ξ x] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\alpha\gamma} & \frac{(-1+e^{\alpha\gamma})\xi}{\alpha\gamma} \\ 0 & 1 \end{pmatrix}$$

```
𝔼[α₁ a + ξ₁ x].𝔼[α₂ a + ξ₂ x] ==
  𝔼[(α₁ + α₂) a + (((α₁ + α₂) ((-1 + e^(γ α₁)) α₂ ξ₁ + e^(γ α₁) (-1 + e^(γ α₂)) α₁ ξ₂)) / ((-1 + e^(γ (α₁+α₂))) α₁ α₂))
     x] // Simplify
```

True

```
𝔼[α₁ a + ξ₁ x].𝔼[α₂ a + ξ₂ x] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\gamma(\alpha_1+\alpha_2)} & \frac{\frac{(-1+e^{\gamma\alpha_1})\xi_1}{\alpha_1}+\frac{e^{\gamma\alpha_1}(-1+e^{\gamma\alpha_2})\xi_2}{\alpha_2}}{\gamma} \\ 0 & 1 \end{pmatrix}$$

```
sol =
 Simplify[{α, ξ} /. Solve[Thread[Flatten /@ (𝔼[α1 a + ξ1 x].𝔼[α2 a + ξ2 x] == 𝔼[α a + ξ x])],
      {α, ξ}][[1]] /. C[_] → 0] /. {α1 → α_i, α2 → α_j, ξ1 → ξ_i, ξ2 → ξ_j}
```

$$\left\{\alpha_i+\alpha_j,\ \left((\alpha_i+\alpha_j)\left(\left(-1+e^{\gamma\alpha_i}\right)\alpha_j\,\xi_i+e^{\gamma\alpha_i}\left(-1+e^{\gamma\alpha_j}\right)\alpha_i\,\xi_j\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i\,\alpha_j\right)\right\}$$

```
Δ_(k_→i_,j_)[ℰ_] := Simplify[ℰ /. {α_k → α_i + α_j,
    ξ_k → ((α_i + α_j) ((-1 + e^(γ α_i)) α_j ξ_i + e^(γ α_i) (-1 + e^(γ α_j)) α_i ξ_j)) / ((-1 + e^(γ (α_i+α_j))) α_i α_j)}]
```

```
α₁ // Δ_(1→1,2) // Δ_(2→2,3)
```

$$\alpha_1+\alpha_2+\alpha_3$$

```
(α₁ // Δ_(1→1,2) // Δ_(2→2,3)) == (α₁ // Δ_(1→2,3) // Δ_(2→1,2))
```

True

```
ξ₁ // Δ_(1→1,2) // Δ_(2→2,3)
```

$$\left((\alpha_1+\alpha_2+\alpha_3)\left(e^{\gamma\alpha_1}\left(-1+e^{\gamma\alpha_2}\right)\alpha_1\,\alpha_3\,\xi_2+\alpha_2\left(\left(-1+e^{\gamma\alpha_1}\right)\alpha_3\,\xi_1+e^{\gamma(\alpha_1+\alpha_2)}\left(-1+e^{\gamma\alpha_3}\right)\alpha_1\,\xi_3\right)\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_1+\alpha_2+\alpha_3)}\right)\alpha_1\,\alpha_2\,\alpha_3\right)$$

```
ξ₁ // Δ_(1→2,3) // Δ_(2→1,2)
```

$$\left((\alpha_1+\alpha_2+\alpha_3)\left(e^{\gamma\alpha_1}\left(-1+e^{\gamma\alpha_2}\right)\alpha_1\,\alpha_3\,\xi_2+\alpha_2\left(\left(-1+e^{\gamma\alpha_1}\right)\alpha_3\,\xi_1+e^{\gamma(\alpha_1+\alpha_2)}\left(-1+e^{\gamma\alpha_3}\right)\alpha_1\,\xi_3\right)\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_1+\alpha_2+\alpha_3)}\right)\alpha_1\,\alpha_2\,\alpha_3\right)$$

```
(ξ₁ // Δ_(1→1,2) // Δ_(2→2,3)) == (ξ₁ // Δ_(1→2,3) // Δ_(2→1,2))
```

True

Simplify[(ξ_1 // $\Delta_{1\to 1,2}$) == (ξ_1 // $\Delta_{1\to 2,1}$)]

$$\left(\left(-1+e^{\gamma \alpha_1}\right)\left(-1+e^{\gamma \alpha_2}\right)(\alpha_1+\alpha_2)(-\alpha_2\,\xi_1+\alpha_1\,\xi_2)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_1+\alpha_2)}\right)\alpha_1\,\alpha_2\right) == 0$$

$\left((\alpha_i+\alpha_j)\left(\left(-1+e^{\gamma \alpha_i}\right)\alpha_j\,\xi_i+e^{\gamma \alpha_i}\left(-1+e^{\gamma \alpha_j}\right)\alpha_i\,\xi_j\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i\,\alpha_j\right)$ // **PowerExpand** // **FullSimplify**

$$\left((\alpha_i+\alpha_j)\left(\left(-1+e^{\gamma \alpha_i}\right)\alpha_j\,\xi_i+e^{\gamma \alpha_i}\left(-1+e^{\gamma \alpha_j}\right)\alpha_i\,\xi_j\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i\,\alpha_j\right)$$

Simplify$\left[\left((\alpha_i+\alpha_j)\left(\left(-1+e^{\gamma \alpha_i}\right)\alpha_j\,\xi_i+e^{\gamma \alpha_i}\left(-1+e^{\gamma \alpha_j}\right)\alpha_i\,\xi_j\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i\,\alpha_j\right) \text{ /. } \xi_j\to 0\right]$

$$\frac{\left(-1+e^{\gamma \alpha_i}\right)(\alpha_i+\alpha_j)\,\xi_i}{\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i}$$

Simplify$\left[\left((\alpha_i+\alpha_j)\left(\left(-1+e^{\gamma \alpha_i}\right)\alpha_j\,\xi_i+e^{\gamma \alpha_i}\left(-1+e^{\gamma \alpha_j}\right)\alpha_i\,\xi_j\right)\right)\Big/\left(\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_i\,\alpha_j\right) \text{ /. } \xi_i\to 0\right]$

$$\frac{e^{\gamma \alpha_i}\left(-1+e^{\gamma \alpha_j}\right)(\alpha_i+\alpha_j)\,\xi_j}{\left(-1+e^{\gamma(\alpha_i+\alpha_j)}\right)\alpha_j}$$