Pensieve header: Products of multiple exponentials in the β-yax algebra; continues BW.nb and pensieve://2017-06/; continued pensieve://2017-08/.

```
ME = MatrixExp; MF = MatrixForm;
```

Representing β-yax

```
y = ( 0  0 ); a = ( 0  0 ); x = ( 0  (e^-β - 1) γ ); b = ( β  0 ); q = e^β;
    (-1  0 )    ( 0 -γ )    ( 0          0        )    ( 0  0 )
MF /@ {t = Simplify[β a - γ b], A = ME[-β a / γ], B = ME[-b], T = ME[t / γ]}
```

$$\left\{\begin{pmatrix} -\beta\gamma & 0 \\ 0 & -\beta\gamma \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta} \end{pmatrix}, \begin{pmatrix} e^{-\beta} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}\right\}$$

```
{a.x - x.a == γ x, x.A == q A.x, a.y - y.a == -γ y, b.y - y.b == -β y, x.y - q y.x == γ ((1 0; 0 1) - T.A.A)} // Simplify
```

```
{True, True, True, True, True}
```

```
P = (1 0; 0 1);
Do[P = Expand[P.ME[η_i y].ME[α_i a].ME[ξ_i x]], {i, 2}];
P // MF
```

$$\begin{pmatrix} 1+\gamma\eta_2\xi_1-e^{-\beta}\gamma\eta_2\xi_1 & -e^{-\gamma\alpha_2}\gamma\xi_1+e^{-\beta-\gamma\alpha_2}\gamma\xi_1-\gamma\xi_2+e^{-\beta}\gamma\xi_2-\gamma^2\eta_2\xi_1\xi \\ -\eta_1-e^{-\gamma\alpha_1}\eta_2-\gamma\eta_1\eta_2\xi_1+e^{-\beta}\gamma\eta_1\eta_2\xi_1 & e^{-\gamma\alpha_1-\gamma\alpha_2}+e^{-\gamma\alpha_2}\gamma\eta_1\xi_1-e^{-\beta-\gamma\alpha_2}\gamma\eta_1\xi_1+\gamma\eta_1\xi_2-e^{-\beta}\gamma\eta_1\xi_2+e^{-\gamma\alpha_1}\gamma\eta_2\xi_2-e^{-\beta-} \end{pmatrix}$$

```
eqn0 = Thread[Flatten /@ (P == τo ME[ηo y].ME[αo a].ME[ξo x])]
```

$$\begin{aligned}
&\{1+\gamma\eta_2\xi_1-e^{-\beta}\gamma\eta_2\xi_1 == \tau o,\\
&-e^{-\gamma\alpha_2}\gamma\xi_1+e^{-\beta-\gamma\alpha_2}\gamma\xi_1-\gamma\xi_2+e^{-\beta}\gamma\xi_2-\gamma^2\eta_2\xi_1\xi_2-e^{-2\beta}\gamma^2\eta_2\xi_1\xi_2+2e^{-\beta}\gamma^2\eta_2\xi_1\xi_2 == -e^{-\beta}\left(-1+e^{\beta}\right)\gamma\xi o\,\tau o,\\
&-\eta_1-e^{-\gamma\alpha_1}\eta_2-\gamma\eta_1\eta_2\xi_1+e^{-\beta}\gamma\eta_1\eta_2\xi_1 == -\eta o\,\tau o,\\
&e^{-\gamma\alpha_1-\gamma\alpha_2}+e^{-\gamma\alpha_2}\gamma\eta_1\xi_1-e^{-\beta-\gamma\alpha_2}\gamma\eta_1\xi_1+\gamma\eta_1\xi_2-e^{-\beta}\gamma\eta_1\xi_2+e^{-\gamma\alpha_1}\gamma\eta_2\xi_2-e^{-\beta-\gamma\alpha_1}\gamma\eta_2\xi_2+\\
&\quad\gamma^2\eta_1\eta_2\xi_1\xi_2+e^{-2\beta}\gamma^2\eta_1\eta_2\xi_1\xi_2-2e^{-\beta}\gamma^2\eta_1\eta_2\xi_1\xi_2 == \left(e^{-\alpha o\gamma}+e^{-\beta}\left(-1+e^{\beta}\right)\gamma\eta o\,\xi o\right)\tau o\}
\end{aligned}$$

```
Solve[eqn0[[1]], τo]
```

$$\left\{\left\{\tau o \to e^{-\beta}\left(e^{\beta}-\gamma\eta_2\xi_1+e^{\beta}\gamma\eta_2\xi_1\right)\right\}\right\}$$

```
P = (1 0; 0 1);
Do[P = Expand[P.ME[η_i y].ME[α_i a].ME[ξ_i x]], {i, 2}];
Column[
 sol = First@Solve[Thread[Flatten /@ (P == τo ME[ηo y].ME[αo a].ME[ξo x])], {τo, ηo, αo, ξo}] /.
   (v_ → ℰ_) :> (v → Simplify@PowerExpand[ℰ])
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\tau o \to 1+e^{-\beta}\left(-1+e^{\beta}\right)\gamma\eta_2\xi_1$$

$$\eta o \to \eta_1+\frac{e^{\beta-\gamma\alpha_1}\eta_2}{e^{\beta}+\left(-1+e^{\beta}\right)\gamma\eta_2\xi_1}$$

$$\alpha o \to \alpha_1+\frac{-2\beta+2\,\text{Log}\left[e^{\beta}+\left(-1+e^{\beta}\right)\gamma\eta_2\xi_1\right]+\gamma\alpha_2}{\gamma}$$

$$\xi o \to \frac{e^{-\gamma\alpha_2}\left(e^{\beta+\gamma\alpha_2}\xi_2+\xi_1\left(e^{\beta}+e^{\gamma\alpha_2}\left(-1+e^{\beta}\right)\gamma\eta_2\xi_2\right)\right)}{e^{\beta}+\left(-1+e^{\beta}\right)\gamma\eta_2\xi_1}$$

```
Column[{fs = FullSimplify[#], TeXForm@fs} & /@ ({ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol)]
```

$$\left\{\eta_1+\frac{e^{\beta-\gamma\alpha_1}\eta_2}{e^\beta+\left(-1+e^\beta\right)\gamma\eta_2\xi_1},\ \text{\frac\{\eta \_2 e\^\{\beta -\alpha \_1 \gamma \}\}\{\left(e\^\{\beta \}-1\right) \gamma \eta \_2 \xi \_1+e\^\{\beta \}\}+\eta \_1}\right\}$$

$$\left\{\frac{2\left(-\beta+\text{Log}\left[e^\beta+\left(-1+e^\beta\right)\gamma\eta_2\xi_1\right]\right)}{\gamma}+\alpha_1+\alpha_2,\ \text{\alpha \_1+\alpha \_2+\frac\{2 \left(\log \left(\left(e\^\{\beta \}-1\right) \gamma \eta \_2 \xi \_1+e\^\{\beta \}\right)-\beta \right)\}\{\gamma \}}\right\}$$

$$\left\{\frac{e^{\beta-\gamma\alpha_2}\xi_1}{e^\beta+\left(-1+e^\beta\right)\gamma\eta_2\xi_1}+\xi_2,\ \text{\frac\{\xi \_1 e\^\{\beta -\alpha \_2 \gamma \}\}\{\left(e\^\{\beta \}-1\right) \gamma \eta \_2 \xi \_1+e\^\{\beta \}\}+\xi \_2}\right\}$$

$$\left\{1+e^{-\beta}\left(-1+e^\beta\right)\gamma\eta_2\xi_1,\ \text{e\^\{-\beta \} \left(e\^\{\beta \}-1\right) \gamma \eta \_2 \xi \_1+1}\right\}$$

$$\left\{-\frac{\text{Log}\left[1+e^{-\beta}\left(-1+e^\beta\right)\gamma\eta_2\xi_1\right]}{\beta\gamma},\ \text{-\frac\{\log \left(e\^\{-\beta \} \left(e\^\{\beta \}-1\right) \gamma \eta \_2 \xi \_1+1\right)\}\{\beta \gamma \}}\right\}$$

```
Series[(e^β + (-1 + e^β) γ η2 ξ1)^-1, {β, 0, 3}]
```

$$1+\left(-1-\gamma\eta_2\xi_1\right)\beta+\left(\frac{1}{2}+\frac{3}{2}\gamma\eta_2\xi_1+\gamma^2\eta_2^2\xi_1^2\right)\beta^2+\left(-\frac{1}{6}-\frac{7}{6}\gamma\eta_2\xi_1-2\gamma^2\eta_2^2\xi_1^2-\gamma^3\eta_2^3\xi_1^3\right)\beta^3+O[\beta]^4$$

```
Column@Series[{ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol, {β, 0, 3}]
```

$$\left(\eta_1+e^{-\gamma\alpha_1}\eta_2\right)-e^{-\gamma\alpha_1}\gamma\eta_2^2\xi_1\beta+\frac{1}{2}e^{-\gamma\alpha_1}\gamma\eta_2^2\xi_1\left(1+2\gamma\eta_2\xi_1\right)\beta^2-\frac{1}{6}\left(e^{-\gamma\alpha_1}\gamma\eta_2^2\xi_1\left(1+6\gamma\eta_2\xi_1+6\gamma^2\eta_2^2\xi_1^2\right)\right)\beta^3+O[\beta]^4$$

$$(\alpha_1+\alpha_2)+2\eta_2\xi_1\beta+\left(-\eta_2\xi_1-\gamma\eta_2^2\xi_1^2\right)\beta^2+\left(\frac{\eta_2\xi_1}{3}+\gamma\eta_2^2\xi_1^2+\frac{2}{3}\gamma^2\eta_2^3\xi_1^3\right)\beta^3+O[\beta]^4$$

$$e^{-\gamma\alpha_2}\left(\xi_1+e^{\gamma\alpha_2}\xi_2\right)-e^{-\gamma\alpha_2}\gamma\eta_2\xi_1^2\beta+\frac{1}{2}e^{-\gamma\alpha_2}\gamma\eta_2\xi_1^2\left(1+2\gamma\eta_2\xi_1\right)\beta^2-\frac{1}{6}\left(e^{-\gamma\alpha_2}\gamma\eta_2\xi_1^2\left(1+6\gamma\eta_2\xi_1+6\gamma^2\eta_2^2\xi_1^2\right)\right)\beta^3+O[\beta]^4$$

$$1+\gamma\eta_2\xi_1\beta-\frac{1}{2}(\gamma\eta_2\xi_1)\beta^2+\frac{1}{6}\gamma\eta_2\xi_1\beta^3+O[\beta]^4$$

$$-\eta_2\xi_1+\frac{1}{2}\left(\eta_2\xi_1+\gamma\eta_2^2\xi_1^2\right)\beta+\frac{1}{6}\left(-\eta_2\xi_1-3\gamma\eta_2^2\xi_1^2-2\gamma^2\eta_2^3\xi_1^3\right)\beta^2+\frac{1}{24}\left(\eta_2\xi_1+7\gamma\eta_2^2\xi_1^2+12\gamma^2\eta_2^3\xi_1^3+6\gamma^3\eta_2^4\xi_1^4\right)\beta^3+O[\beta]^4$$

```
Column[{fs = Collect[Normal@Series[#, {β, 0, 2}], β, FullSimplify], TeXForm@fs} & /@
  ({ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol)]
```

$$\left\{\eta_1+e^{-\gamma\alpha_1}\eta_2-e^{-\gamma\alpha_1}\beta\gamma\eta_2^2\xi_1+\frac{1}{2}e^{-\gamma\alpha_1}\beta^2\gamma\eta_2^2\xi_1\left(1+2\gamma\eta_2\xi_1\right),\ \text{\frac\{1\}\{2\} \beta \^2 \gamma \eta \_2\^2 \xi \_1 e\^\{\alpha \_1 (-\gamma)\} \left(2 \gamma \eta \_2 \xi \_1+1\right)+\beta \gamma \eta \_2\^2 \xi \_1 \left(-e\^\{\alpha \_1 (-\gamma)\}\right)+\eta \_2 e\^\{\alpha \_1 (-\gamma)\}+\eta \_1}\right\}$$

$$\left\{\alpha_1+\alpha_2+2\beta\eta_2\xi_1-\beta^2\eta_2\xi_1\left(1+\gamma\eta_2\xi_1\right),\ \text{\alpha \_1+\alpha \_2+\beta \^2 \left(-\eta \_2\right) \xi \_1 \left(\gamma \eta \_2 \xi \_1+1\right)+2 \beta \eta \_2 \xi \_1}\right\}$$

$$\left\{e^{-\gamma\alpha_2}\xi_1-e^{-\gamma\alpha_2}\beta\gamma\eta_2\xi_1^2+\frac{1}{2}e^{-\gamma\alpha_2}\beta^2\gamma\eta_2\xi_1^2\left(1+2\gamma\eta_2\xi_1\right)+\xi_2,\ \text{\frac\{1\}\{2\} \beta \^2 \gamma \eta \_2 \xi \_1\^2 e\^\{\alpha \_2 (-\gamma)\} \left(2 \gamma \eta \_2 \xi \_1+1\right)+\beta \gamma \eta \_2 \xi \_1\^2 \left(-e\^\{\alpha \_2 (-\gamma)\}\right)+\xi \_1 e\^\{\alpha \_2 (-\gamma)\}+\xi \_2}\right\}$$

$$\left\{1+\beta\gamma\eta_2\xi_1-\frac{1}{2}\beta^2\gamma\eta_2\xi_1,\ \text{-\frac\{1\}\{2\} \beta \^2 \gamma \eta \_2 \xi \_1+\beta \gamma \eta \_2 \xi \_1+1}\right\}$$

$$\left\{-\eta_2\xi_1+\frac{1}{2}\beta\eta_2\xi_1\left(1+\gamma\eta_2\xi_1\right)-\frac{1}{6}\beta^2\eta_2\xi_1\left(1+\gamma\eta_2\xi_1\right)\left(1+2\gamma\eta_2\xi_1\right),\ \text{-\frac\{1\}\{6\} \beta \^2 \eta \_2 \xi \_1 \left(\gamma \eta \_2 \xi \_1+1\right) \left(2 \gamma \eta \_2 \xi \_1+1\right)+\frac\{1\}\{2\} \beta \eta \_2 \xi \_1 \left(\gamma \eta \_2 \xi \_1+1\right)-\eta \_2 \xi \_1}\right\}$$

```
P = {{1, 0}, {0, 1}};
Do[P = Expand[P.ME[η_i y].ME[α_i a].ME[ξ_i x]], {i, 3}];
Column[
 First@Solve[Thread[Flatten /@ (P == τo ME[ηo y].ME[αo a].ME[ξo x])], {τo, ηo, αo, ξo}]
]
```

$Aborted

xy to yax

```
Column[
 sol1 =
  First@Solve[Thread[Flatten /@ (ME[ξ x].ME[η y] == τo ME[ηo y].ME[αo a].ME[ξo x])], {τo, ηo, αo, ξo}] /.
   (v_ → ℰ_) :> (v → FullSimplify@PowerExpand[ℰ])
]
```

... Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\tau o \to 1 + e^{-\beta} \left(-1 + e^{\beta}\right) \gamma \eta \xi$$

$$\eta o \to \frac{e^{\beta} \eta}{e^{\beta} - \gamma \eta \xi + e^{\beta} \gamma \eta \xi}$$

$$\alpha o \to \frac{2 \left(-\beta + \text{Log}\left[e^{\beta} - \gamma \eta \xi + e^{\beta} \gamma \eta \xi\right]\right)}{\gamma}$$

$$\xi o \to \frac{e^{\beta} \xi}{e^{\beta} - \gamma \eta \xi + e^{\beta} \gamma \eta \xi}$$

```
Column@Series[{ηo, αo, ξo, τo, Log[τo]/(-β γ)} /. sol1, {β, 0, 2}]
```

$$\eta - \gamma \eta^2 \xi \beta + \left(\frac{1}{2} \gamma \eta^2 \xi + \gamma^2 \eta^3 \xi^2\right) \beta^2 + O[\beta]^3$$

$$2 \eta \xi \beta + \left(-\eta \xi - \gamma \eta^2 \xi^2\right) \beta^2 + O[\beta]^3$$

$$\xi - \gamma \eta \xi^2 \beta + \left(\frac{1}{2} \gamma \eta \xi^2 + \gamma^2 \eta^2 \xi^3\right) \beta^2 + O[\beta]^3$$

$$1 + \gamma \eta \xi \beta - \frac{1}{2} (\gamma \eta \xi) \beta^2 + O[\beta]^3$$

$$-\eta \xi + \frac{1}{2} \left(\eta \xi + \gamma \eta^2 \xi^2\right) \beta + \frac{1}{6} \left(-\eta \xi - 3 \gamma \eta^2 \xi^2 - 2 \gamma^2 \eta^3 \xi^3\right) \beta^2 + O[\beta]^3$$