Pensieve header: The Benkart-Witherspoon representation; continues pensieve://2017-06/.

$$\rho = 0;\ \sigma = \beta;\ \left\{\hbar = \frac{\sigma-\rho}{\gamma\beta},\ q = e^{\hbar\gamma\beta},\ -\hbar\gamma,\ -\hbar\beta\right\}$$

$$\left\{\frac{1}{\gamma},\ e^{\beta},\ -1,\ -\frac{\beta}{\gamma}\right\}$$

$$y = \begin{pmatrix} 0 & 0 \\ -e^{\rho} & 0 \end{pmatrix};\ a = \frac{\gamma}{\rho-\sigma}\begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix};\ x = \frac{e^{-\sigma}-e^{-\rho}}{\hbar}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ b = \frac{\beta}{\sigma-\rho}\begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix};$$

```
ME = MatrixExp; MF = MatrixForm;
MF /@ {y, a, x, b, t = Simplify[β a - γ b], A = ME[-ℏ β a], B = ME[-ℏ γ b], T = ME[ℏ t]}
```

$$\left\{\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix}, \begin{pmatrix} 0 & (-1+e^{-\beta})\gamma \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\beta\gamma & 0 \\ 0 & -\beta\gamma \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta} \end{pmatrix}, \begin{pmatrix} e^{-\beta} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{-\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}\right\}$$

$$\left\{a.x - x.a == \gamma x,\ x.A == q A.x,\ a.y - y.a == -\gamma y,\ b.y - y.b == -\beta y,\ x.y - q y.x == \frac{1}{\hbar}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - T.A.A\right)\right\} \text{ // Simplify}$$

```
{True, True, True, True, True}
```

```
ME[ξ x].ME[η y] // Simplify // MF
```

$$\begin{pmatrix} 1 + e^{-\beta}(-1+e^{\beta})\gamma\eta\xi & (-1+e^{-\beta})\gamma\xi \\ -\eta & 1 \end{pmatrix}$$

```
ME[η y].ME[ξ x] // Simplify // MF
```

$$\begin{pmatrix} 1 & (-1+e^{-\beta})\gamma\xi \\ -\eta & 1 + e^{-\beta}(-1+e^{\beta})\gamma\eta\xi \end{pmatrix}$$

```
ME[ξ x].ME[η y].ME[-ξ x].ME[-η y] // Simplify // MF
```

$$\begin{pmatrix} 1 + e^{-\beta}(-1+e^{\beta})\gamma\eta\xi + e^{-2\beta}(-1+e^{\beta})^2\gamma^2\eta^2\xi^2 & e^{-2\beta}(-1+e^{\beta})^2\gamma^2\eta\xi^2 \\ -e^{-\beta}(-1+e^{\beta})\gamma\eta^2\xi & 1 + (-1+e^{-\beta})\gamma\eta\xi \end{pmatrix}$$

```
x.y - y.x // MF
```

$$\begin{pmatrix} -(-1+e^{-\beta})\gamma & 0 \\ 0 & (-1+e^{-\beta})\gamma \end{pmatrix}$$

```
eqn0 = ME[ξ0 x].ME[η0 y] == ME[η y].ME[α a].ME[ξ x].ME[τ t]
```

$$\{\{1 + e^{-\beta}(-1+e^{\beta})\gamma\eta 0\,\xi 0,\ -e^{-\beta}(-1+e^{\beta})\gamma\xi 0\},\ \{-\eta 0,\ 1\}\} ==$$
$$\{\{e^{-\beta\gamma\tau},\ -e^{-\beta-\beta\gamma\tau}(-1+e^{\beta})\gamma\xi\},\ \{-e^{-\beta\gamma\tau}\eta,\ e^{-\beta\gamma\tau}(e^{-\alpha\gamma} + e^{-\beta}(-1+e^{\beta})\gamma\eta\xi)\}\}$$

```
MF /@ eqn0
```

$$\begin{pmatrix} 1 + e^{-\beta}(-1+e^{\beta})\gamma\eta 0\,\xi 0 & -e^{-\beta}(-1+e^{\beta})\gamma\xi 0 \\ -\eta 0 & 1 \end{pmatrix} == \begin{pmatrix} e^{-\beta\gamma\tau} & -e^{-\beta-\beta\gamma\tau}(-1+e^{\beta})\gamma\xi \\ -e^{-\beta\gamma\tau}\eta & e^{-\beta\gamma\tau}(e^{-\alpha\gamma} + e^{-\beta}(-1+e^{\beta})\gamma\eta\xi) \end{pmatrix}$$

$$\{\text{sol0}\} = \text{Solve}\left[1 + e^{-\beta}(-1+e^{\beta})\gamma\eta 0\,\xi 0 == e^{-\beta\gamma\tau},\ \tau\right] \text{ /. } C[1] \to 0$$

$$\left\{\left\{\tau \to \frac{\text{Log}\left[\frac{e^{\beta}}{e^{\beta} - \gamma\eta 0\,\xi 0 + e^{\beta}\gamma\eta 0\,\xi 0}\right]}{\beta\gamma}\right\}\right\}$$

```
MF /@ (eqn1 = Simplify /@ (eqn0 /. sol0))
```

$$\begin{pmatrix} 1+e^{-\beta}\left(-1+e^{\beta}\right)\gamma\,\eta0\,\xi0 & \left(-1+e^{-\beta}\right)\gamma\,\xi0 \\ -\eta0 & 1 \end{pmatrix} == \begin{pmatrix} 1+e^{-\beta}\left(-1+e^{\beta}\right)\gamma\,\eta0\,\xi0 & -e^{-2\beta}\left(-1+e^{\beta}\right)\gamma\,\xi\left(-\gamma\,\eta0\,\xi0+e^{\beta}\left(1+\gamma\,\eta0\,\xi0\right)\right) \\ \eta\left(-1+\left(-1+e^{-\beta}\right)\gamma\,\eta0\,\xi0\right) & e^{-\beta}\left(e^{-\alpha\gamma}+\gamma\,\eta\,\xi-e^{-\beta}\gamma\,\eta\,\xi\right)\left(-\gamma\,\eta0\,\xi0+e^{\beta}\left(1+\gamma\,\eta0\,\xi0\right)\right) \end{pmatrix}$$

```
{sol1} = Solve[Thread[Flatten /@ eqn1], {η, α, ξ}]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\left\{\left\{\eta\to\frac{e^{\beta}\,\eta0}{e^{\beta}-\gamma\,\eta0\,\xi0+e^{\beta}\gamma\,\eta0\,\xi0},\ \alpha\to\frac{\mathrm{Log}\left[e^{-2\beta}\left(e^{\beta}-\gamma\,\eta0\,\xi0+e^{\beta}\gamma\,\eta0\,\xi0\right)^2\right]}{\gamma},\ \xi\to\frac{e^{\beta}\,\xi0}{e^{\beta}-\gamma\,\eta0\,\xi0+e^{\beta}\gamma\,\eta0\,\xi0}\right\}\right\}$$

```
Column@Series[{η, α, ξ, τ} /. sol1 /. sol0, {β, 0, 2}]
```

$$\eta0-\gamma\,\eta0^2\,\xi0\,\beta+\left(\frac{1}{2}\gamma\,\eta0^2\,\xi0+\gamma^2\,\eta0^3\,\xi0^2\right)\beta^2+O[\beta]^3$$
$$2\,\eta0\,\xi0\,\beta+\left(-\eta0\,\xi0-\gamma\,\eta0^2\,\xi0^2\right)\beta^2+O[\beta]^3$$
$$\xi0-\gamma\,\eta0\,\xi0^2\,\beta+\left(\frac{1}{2}\gamma\,\eta0\,\xi0^2+\gamma^2\,\eta0^2\,\xi0^3\right)\beta^2+O[\beta]^3$$
$$-\eta0\,\xi0+\frac{1}{2}\left(\eta0\,\xi0+\gamma\,\eta0^2\,\xi0^2\right)\beta+\frac{1}{6}\left(-\eta0\,\xi0-3\,\gamma\,\eta0^2\,\xi0^2-2\,\gamma^2\,\eta0^3\,\xi0^3\right)\beta^2+O[\beta]^3$$