Pensieve header: Products of multiple exponentials in the yax algebra; verifies Monoblog/170702.

Representing yax

```
ρ1 = {{0, 0, 1}, {0, 0, 0}, {0, 0, 0}}; ρy = {{0, 0, 0}, {0, 0, τ}, {0, 0, 0}}; ρa = {{0, 0, 0}, {0, -α, 0}, {0, 0, 0}}; ρx = {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}}; ρ0 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}};
B[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
```

```
{B[ρa, ρx] == α ρx, B[ρa, ρy] == -α ρy, B[ρx, ρy] == τ ρ1, B[ρ1, ρy] == ρ0, B[ρ1, ρa] == ρ0, B[ρ1, ρx] == ρ0}
```

```
{True, True, True, True, True, True}
```

```
P = IdentityMatrix[3];
Do[
  P = Expand[P.MatrixExp[η_i ρy].MatrixExp[α_i ρa].MatrixExp[ξ_i ρx]],
  {i, 1, 5}
 ];
P // MatrixForm
```

$$\begin{pmatrix} 1 & e^{-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5}\xi_1+e^{-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5}\xi_2+e^{-\alpha\alpha_4-\alpha\alpha_5}\xi_3+e^{-\alpha\alpha_5}\xi_4+\xi_5 & \tau\eta_2\xi_1+e^{-\alpha\alpha_2}\tau\eta_3\xi_1+e^{-\alpha\alpha_2-\alpha\alpha_3}\tau\eta_4\xi_1+e^{-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4}\tau\eta_5\xi_1+\tau\eta_3\xi_2+e^{-\alpha\alpha_3}\tau\eta_4\xi_2+e^{-\alpha\alpha_3-\alpha\alpha_4}\tau\eta_5\xi_2+\tau\eta_4\xi_3+e^{-\alpha\alpha_4}\tau\eta_5\xi_3+\tau\eta_5\xi_4 \\ 0 & e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5} & \tau\eta_1+e^{-\alpha\alpha_1}\tau\eta_2+e^{-\alpha\alpha_1-\alpha\alpha_2}\tau\eta_3+e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3}\tau\eta_4+e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4}\tau\eta_5 \\ 0 & 0 & e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5} \end{pmatrix}$$

```
Column[
 First@Solve[Thread[Flatten /@ (P == MatrixExp[Y ρy].MatrixExp[A ρa].MatrixExp[X ρx].MatrixExp[σ ρ1])],
    {Y, A, X, σ}] /. (ν_ → ℰ_) :> (ν → Expand@PowerExpand[ℰ])
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$\begin{aligned}
&Y \to \eta_1+e^{-\alpha\alpha_1}\eta_2+e^{-\alpha\alpha_1-\alpha\alpha_2}\eta_3+e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3}\eta_4+e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4}\eta_5\\
&A \to \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\\
&X \to e^{-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5}\xi_1+e^{-\alpha\alpha_3-\alpha\alpha_4-\alpha\alpha_5}\xi_2+e^{-\alpha\alpha_4-\alpha\alpha_5}\xi_3+e^{-\alpha\alpha_5}\xi_4+\xi_5\\
&\sigma \to \tau\eta_2\xi_1+e^{-\alpha\alpha_2}\tau\eta_3\xi_1+e^{-\alpha\alpha_2-\alpha\alpha_3}\tau\eta_4\xi_1+e^{-\alpha\alpha_2-\alpha\alpha_3-\alpha\alpha_4}\tau\eta_5\xi_1+\\
&\quad\tau\eta_3\xi_2+e^{-\alpha\alpha_3}\tau\eta_4\xi_2+e^{-\alpha\alpha_3-\alpha\alpha_4}\tau\eta_5\xi_2+\tau\eta_4\xi_3+e^{-\alpha\alpha_4}\tau\eta_5\xi_3+\tau\eta_5\xi_4
\end{aligned}$$

```
PS = IdentityMatrix[3];
Do[
  PS = Expand[PS.MatrixExp[η_i ρy + α_i ρa + ξ_i ρx]],
  {i, 1, 3}
 ];
PS // MatrixForm
```

$$\begin{pmatrix} 1 & \frac{e^{-\alpha\alpha_2-\alpha\alpha_3}\xi_1}{\alpha\alpha_1}-\frac{e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3}\xi_1}{\alpha\alpha_1}+\frac{e^{-\alpha\alpha_3}\xi_2}{\alpha\alpha_2}-\frac{e^{-\alpha\alpha_2-\alpha\alpha_3}\xi_2}{\alpha\alpha_2}+\frac{\xi_3}{\alpha\alpha_3}-\frac{e^{-\alpha\alpha_3}\xi_3}{\alpha\alpha_3} & -\frac{\tau\eta_1\xi_1}{\alpha^2\alpha_1^2}+\frac{e^{-\alpha\alpha_1}\tau\eta_1\xi_1}{\alpha^2\alpha_1^2}+\frac{\tau\eta_1\xi_1}{\alpha\alpha_1}+\frac{\tau\eta_2\xi_1}{\alpha^2\alpha_1\alpha_2}-\frac{e^{-\alpha\alpha_1}\tau\eta_2\xi_1}{\alpha^2\alpha_1\alpha_2}-\frac{e^{-\alpha\alpha_2}\tau\eta_2\xi_1}{\alpha^2\alpha_1\alpha_2}+\frac{e} \\ 0 & e^{-\alpha\alpha_1-\alpha\alpha_2-\alpha\alpha_3} & \\ 0 & 0 & \end{pmatrix}$$

```
Column[
 First@Solve[Thread[Flatten /@ (PS == MatrixExp[Y ρy].MatrixExp[A ρa].MatrixExp[X ρx].MatrixExp[σ ρ1])],
    {Y, A, X, σ}] /. (v_ → ℰ_) :> (v → Expand@PowerExpand[ℰ])
]
```

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$Y \to \frac{\eta_1}{\alpha\,\alpha_1} - \frac{e^{-\alpha\,\alpha_1}\,\eta_1}{\alpha\,\alpha_1} + \frac{e^{-\alpha\,\alpha_1}\,\eta_2}{\alpha\,\alpha_2} - \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2}\,\eta_2}{\alpha\,\alpha_2} + \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2}\,\eta_3}{\alpha\,\alpha_3} - \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\eta_3}{\alpha\,\alpha_3}$$

$$A \to \alpha_1 + \alpha_2 + \alpha_3$$

$$X \to \frac{e^{-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\xi_1}{\alpha\,\alpha_1} - \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\xi_1}{\alpha\,\alpha_1} + \frac{e^{-\alpha\,\alpha_3}\,\xi_2}{\alpha\,\alpha_2} - \frac{e^{-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\xi_2}{\alpha\,\alpha_2} + \frac{\xi_3}{\alpha\,\alpha_3} - \frac{e^{-\alpha\,\alpha_3}\,\xi_3}{\alpha\,\alpha_3}$$

$$\sigma \to -\frac{\tau\,\eta_1\,\xi_1}{\alpha^2\,\alpha_1^2} + \frac{e^{-\alpha\,\alpha_1}\,\tau\,\eta_1\,\xi_1}{\alpha^2\,\alpha_1^2} + \frac{\tau\,\eta_1\,\xi_1}{\alpha\,\alpha_1} + \frac{\tau\,\eta_2\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_2} - \frac{e^{-\alpha\,\alpha_1}\,\tau\,\eta_2\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_2} - \frac{e^{-\alpha\,\alpha_2}\,\tau\,\eta_2\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_2} + \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2}\,\tau\,\eta_2\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_2} + \frac{e^{-\alpha\,\alpha_2}\,\tau\,\eta_3\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_3} - \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2}\,\tau\,\eta_3\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_3} - \frac{e^{-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\tau\,\eta_3\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_3} + \frac{e^{-\alpha\,\alpha_1-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\tau\,\eta_3\,\xi_1}{\alpha^2\,\alpha_1\,\alpha_3} - \frac{\tau\,\eta_2\,\xi_2}{\alpha^2\,\alpha_2^2} + \frac{e^{-\alpha\,\alpha_2}\,\tau\,\eta_2\,\xi_2}{\alpha^2\,\alpha_2^2} + \frac{\tau\,\eta_2\,\xi_2}{\alpha\,\alpha_2} + \frac{\tau\,\eta_3\,\xi_2}{\alpha^2\,\alpha_2\,\alpha_3} - \frac{e^{-\alpha\,\alpha_2}\,\tau\,\eta_3\,\xi_2}{\alpha^2\,\alpha_2\,\alpha_3} - \frac{e^{-\alpha\,\alpha_3}\,\tau\,\eta_3\,\xi_2}{\alpha^2\,\alpha_2\,\alpha_3} + \frac{e^{-\alpha\,\alpha_2-\alpha\,\alpha_3}\,\tau\,\eta_3\,\xi_2}{\alpha^2\,\alpha_2\,\alpha_3} - \frac{\tau\,\eta_3\,\xi_3}{\alpha^2\,\alpha_3^2} + \frac{e^{-\alpha\,\alpha_3}\,\tau\,\eta_3\,\xi_3}{\alpha^2\,\alpha_3^2} + \frac{\tau\,\eta_3\,\xi_3}{\alpha\,\alpha_3}$$

```
Column[
 First@Solve[Thread[Flatten /@ (PS == MatrixExp[Y ρy + A ρa + X ρx + σ ρ1])], {Y, A, X, σ}] /.
   (v_ → ℰ_) :> (v → Simplify@PowerExpand[ℰ])
]
```

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is

$$-\text{Log}\left[e^{\alpha\,(\text{A-Subscript[≪2≫]-Subscript[≪2≫]-Subscript[≪2≫]})}\right] - \text{Log}\left[e^{-\alpha\,\alpha_1-\alpha\,\alpha_2-\alpha\,\alpha_3}\right] - \alpha\,\alpha_1 - \alpha\,\alpha_2 - \alpha\,\alpha_3 == 0.$$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

$$Y \to \left((\alpha_1+\alpha_2+\alpha_3)\left(e^{\alpha\,\alpha_3}\left(-1+e^{\alpha\,\alpha_2}\right)\alpha_1\,\alpha_3\,\eta_2 + \alpha_2\left(e^{\alpha\,(\alpha_2+\alpha_3)}\left(-1+e^{\alpha\,\alpha_1}\right)\alpha_3\,\eta_1 + \left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1\,\eta_3\right)\right)\right) \Big/ \left(\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)\alpha_1\,\alpha_2\,\alpha_3\right)$$

$$A \to \alpha_1 + \alpha_2 + \alpha_3$$

$$X \to \left((\alpha_1+\alpha_2+\alpha_3)\left(e^{\alpha\,\alpha_1}\left(-1+e^{\alpha\,\alpha_2}\right)\alpha_1\,\alpha_3\,\xi_2 + \alpha_2\left(\left(-1+e^{\alpha\,\alpha_1}\right)\alpha_3\,\xi_1 + e^{\alpha\,(\alpha_1+\alpha_2)}\left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1\,\xi_3\right)\right)\right) \Big/ \left(\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)\alpha_1\,\alpha_2\,\alpha_3\right)$$

$$\sigma \to -\frac{1}{\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)^2\alpha^2\,\alpha_1^2\,\alpha_2^2\,\alpha_3^2}$$
$$\tau\left(\left(-1+e^{\alpha\,\alpha_2}\right)\alpha_1^2\,\alpha_3^2\left(\left(-1+e^{\alpha\,(\alpha_1+\alpha_3)}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)+e^{\alpha\,(\alpha_1+\alpha_3)}\left(-1+e^{\alpha\,\alpha_2}\right)\alpha\,\alpha_1+e^{\alpha\,(\alpha_1+\alpha_3)}\left(-1+e^{\alpha\,\alpha_2}\right)\alpha\,\alpha_3\right)\eta_2\,\xi_2+\right.$$
$$\alpha\,\alpha_2^3\left(e^{\alpha\,(\alpha_2+\alpha_3)}\left(-1+e^{\alpha\,\alpha_1}\right)\alpha_3\,\eta_1+\left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1\,\eta_3\right)\left(\left(-1+e^{\alpha\,\alpha_1}\right)\alpha_3\,\xi_1+e^{\alpha\,(\alpha_1+\alpha_2)}\left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1\,\xi_3\right)+$$
$$\alpha_1\,\alpha_2\,\alpha_3\left(e^{\alpha\,\alpha_3}\left(-1+e^{\alpha\,\alpha_1}\right)\left(-1+e^{\alpha\,\alpha_2}\right)\alpha\,\alpha_2^3\left(\eta_2\,\xi_1+e^{\alpha\,(\alpha_1+\alpha_2)}\,\eta_1\,\xi_2\right)+\right.$$
$$\left(-1+e^{\alpha\,\alpha_2}\right)\left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1\left(-e^{\alpha\,\alpha_1}\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}-\alpha\,\alpha_1\right)\eta_3\,\xi_2+\left(1-e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\,\alpha\,\alpha_1\right)\eta_2\,\xi_3\right)+$$
$$\alpha_3\left(\left(-1+e^{\alpha\,\alpha_2}\right)\left(\left(-1+e^{\alpha\,\alpha_1}\right)\left(1-e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\,\alpha\,\alpha_1\right)\eta_1+e^{\alpha\,\alpha_1}\left(-1+e^{\alpha\,\alpha_3}\right)\alpha\,\alpha_1\,\eta_3\right)\xi_2+\right.$$
$$\eta_2\left(-e^{\alpha\,\alpha_3}\left(-1+e^{\alpha\,\alpha_1}\right)\left(-1+e^{\alpha\,\alpha_2}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}-\alpha\,\alpha_1\right)\xi_1+\alpha\,\alpha_1\right.$$
$$\left.\left.\left.\left.\left.\left(-\left(-1+e^{\alpha\,(\alpha_1+\alpha_3)}\right)\left(-1+e^{\alpha\,(\alpha_1+2\,\alpha_2+\alpha_3)}\right)\xi_2+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\left(-1+e^{\alpha\,\alpha_2}\right)\left(-1+e^{\alpha\,\alpha_3}\right)\xi_3\right)\right)\right)\right)+\right.$$
$$\alpha_2^2\left(e^{\alpha\,(\alpha_2+\alpha_3)}\left(-1+e^{\alpha\,\alpha_1}\right)^2\alpha\,\alpha_3^3\,\eta_1\,\xi_1-\left(-1+e^{\alpha\,\alpha_3}\right)\alpha_1^2\left(-\left(-1+e^{\alpha\,(\alpha_1+\alpha_2)}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)-\right.\right.$$
$$\left.e^{\alpha\,(\alpha_1+\alpha_2)}\left(-1+e^{\alpha\,\alpha_3}\right)\alpha\,\alpha_1\right)\eta_3\,\xi_3+\alpha_3^2\left(\left(-1+e^{\alpha\,\alpha_1}\right)\alpha\,\alpha_1\left(e^{\alpha\,\alpha_3}\left(-1+e^{\alpha\,\alpha_2}\right)\eta_2+\left(-1+e^{\alpha\,\alpha_3}\right)\eta_3\right)\xi_1+\right.$$
$$\eta_1\left(-\left(-1+e^{\alpha\,(\alpha_2+\alpha_3)}\right)\left(-\left(-1+e^{\alpha\,\alpha_1}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\right)+\left(-1+e^{\alpha\,(2\,\alpha_1+\alpha_2+\alpha_3)}\right)\alpha\,\alpha_1\right)\right.$$
$$\left.\left.\left.\xi_1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\left(-1+e^{\alpha\,\alpha_1}\right)\alpha\,\alpha_1\left(\left(-1+e^{\alpha\,\alpha_2}\right)\xi_2+e^{\alpha\,\alpha_2}\left(-1+e^{\alpha\,\alpha_3}\right)\xi_3\right)\right)\right)+$$
$$\alpha_1\,\alpha_3\left(e^{\alpha\,\alpha_2}\left(-1+e^{\alpha\,\alpha_3}\right)\left(\left(-1+e^{\alpha\,\alpha_1}\right)\left(1-e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}\,\alpha\,\alpha_1\right)\eta_1+e^{\alpha\,(\alpha_1+\alpha_3)}\left(-1+e^{\alpha\,\alpha_2}\right)\alpha\,\alpha_1\,\eta_2\right)\xi_3+\right.$$
$$\eta_3\left(-\left(-1+e^{\alpha\,\alpha_1}\right)\left(-1+e^{\alpha\,\alpha_3}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+\alpha_3)}-\alpha\,\alpha_1\right)\xi_1+\alpha\,\alpha_1\right.$$
$$\left.\left.\left.\left.\left(e^{\alpha\,\alpha_1}\left(-1+e^{\alpha\,\alpha_2}\right)\left(-1+e^{\alpha\,\alpha_3}\right)\xi_2-\left(-1+e^{\alpha\,(\alpha_1+\alpha_2)}\right)\left(-1+e^{\alpha\,(\alpha_1+\alpha_2+2\,\alpha_3)}\right)\xi_3\right)\right)\right)\right)\right)$$