Old Quesne.

Problem. Write $\sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$ at $q = e^{\epsilon}$ as a power series in ϵ.

```
Simplify[∏_{k=1}^{n} Series[k (q - 1)/(q^k - 1) /. q → ⅇ^ϵ, {ϵ, 0, 5}]]
```

$$\prod_{k=1}^{n}\left(1+\left(\frac{1}{2}-\frac{k}{2}\right)\epsilon+\frac{1}{12}\left(2-3k+k^2\right)\epsilon^2+\frac{1}{24}\left(1-2k+k^2\right)\epsilon^3+\frac{1}{720}\left(6-15k+10k^2-k^4\right)\epsilon^4+\frac{\left(2-6k+5k^2-k^4\right)\epsilon^5}{1440}+O[\epsilon]^6\right)$$

First find $G_d[n]$, the coefficient of ϵ^d in the above product.

```
G_0[_] = 1;
G_d_[n_] := Simplify@Sum[
    Sum[G_{d-j}[m - 1] SeriesCoefficient[m (q - 1)/(q^m - 1) /. q → ⅇ^ϵ, {ϵ, 0, j}], {m, 1, n}],
    {j, 1, d}
   ];
```

```
Table[G_d[n], {d, 0, 5}]
```

$$\left\{1, -\frac{1}{4}(-1+n)\,n, \frac{1}{288}n\left(10+3n-22n^2+9n^3\right), -\frac{(-1+n)^2 n^2\left(-10-7n+3n^2\right)}{1152}, \frac{1}{4147200}n\left(-1488+2500n+1980n^2-11405n^3+6888n^4+5350n^5-4500n^6+675n^7\right), -\frac{1}{16588800}(-1+n)^2 n^2\left(1488-1012n+8n^2+4153n^3+505n^4-1005n^5+135n^6\right)\right\}$$

```
Q_d_[x_] := Sum[G_d[n] x^n/n!, {n, 0, ∞}]
```

```
Table[Q_d[x], {d, 0, 5}]
```

$$\left\{e^x, -\frac{1}{4}e^x x^2, \frac{1}{288}e^x x^3(32+9x), -\frac{e^x x^2\left(-24+72x^2+32x^3+3x^4\right)}{1152}, \frac{1}{4147200}e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right), -\frac{1}{16588800}e^x x^2\left(34560-518400x^2-153600x^3+450000x^4+281088x^5+58000x^6+4800x^7+135x^8\right)\right\}$$

Comparing with Quesne’s arXiv:math-ph/0305003:

```
c_k_[q_] := (1 - q)^k / (k (1 - q^k));
With[{n = 5}, Collect[
  ⅇ^-x Normal@Series[Exp[Sum[c_k[ⅇ^ϵ] x^k, {k, 1, n + 1}]], {ϵ, 0, n}],
  ϵ, Simplify
 ]]
```

$$1 - \frac{x^2 \epsilon}{4} + \frac{1}{288} x^3 (32 + 9x) \epsilon^2 - \frac{x^2 (-24 + 72x^2 + 32x^3 + 3x^4) \epsilon^3}{1152} + \left(-\frac{x^3}{36} - \frac{x^4}{192} + \frac{x^5}{25} + \frac{113 x^6}{5184} + \frac{x^7}{288} + \frac{x^8}{6144}\right) \epsilon^4 - \frac{1}{16\,588\,800} x^2 (34\,560 - 518\,400 x^2 - 153\,600 x^3 + 450\,000 x^4 + 281\,088 x^5 + 58\,000 x^6 + 4800 x^7 + 135 x^8) \epsilon^5$$

Inverse Quesne.

Problem. Write $\sum_{n=0}^{\infty} \frac{[n]_q! \, x^n}{(n!)^2}$ at $q = e^{\epsilon}$ as a power series in ϵ.

```
Simplify[x^n ∏_{k=1}^{n} Series[(q^k - 1)/(k^2 (q - 1)) /. q → ⅇ^ϵ, {ϵ, 0, 4}]]
```

$$x^n \prod_{k=1}^{n} \left(\frac{1}{k} + \frac{(-1+k)\epsilon}{2k} + \left(-\frac{1}{4} + \frac{1}{12k} + \frac{k}{6}\right)\epsilon^2 + \frac{1}{24}(1 - 2k + k^2)\epsilon^3 + \frac{(-1 + 10k^2 - 15k^3 + 6k^4)\epsilon^4}{720k} + O[\epsilon]^5\right)$$

```
Simplify[∏_{k=1}^{n} Series[(q^k - 1)/(k (q - 1)) /. q → ⅇ^ϵ, {ϵ, 0, 5}]]
```

$$\prod_{k=1}^{n} \left(1 + \frac{1}{2}(-1+k)\epsilon + \frac{1}{12}(1 - 3k + 2k^2)\epsilon^2 + \frac{1}{24}(k - 2k^2 + k^3)\epsilon^3 + \frac{1}{720}(-1 + 10k^2 - 15k^3 + 6k^4)\epsilon^4 + \frac{(-k + 5k^3 - 6k^4 + 2k^5)\epsilon^5}{1440} + O[\epsilon]^6\right)$$

First find $F_d[n]$, the coefficient of ϵ^d in the above product.

```
SeriesCoefficient[(q^k - 1)/(k (q - 1)) /. q → ⅇ^ϵ, {ϵ, 0, 5}]
```

$$\frac{-k + 5k^3 - 6k^4 + 2k^5}{1440}$$

```
Sum[(-k + 5 k^3 - 6 k^4 + 2 k^5)/1440, {k, n1, n2}]
```

$$-\frac{1}{86\,400}(-1 + n1 - n2)(-42\,n1 - 77\,n1^2 + 193\,n1^3 - 112\,n1^4 + 20\,n1^5 - 18\,n2 + 24\,n1\,n2 + 101\,n1^2\,n2 - 92\,n1^3\,n2 + 20\,n1^4\,n2 + 53\,n2^2 + 29\,n1\,n2^2 - 72\,n1^2\,n2^2 + 20\,n1^3\,n2^2 - 23\,n2^3 - 52\,n1\,n2^3 + 20\,n1^2\,n2^3 - 32\,n2^4 + 20\,n1\,n2^4 + 20\,n2^5)$$

```
F0[_] = 1;
Fd_[n_] := Simplify@Sum[
   Sum[F(d-j)[m - 1] SeriesCoefficient[(q^m - 1)/(m (q - 1)) /. q → ⅇ^ϵ, {ϵ, 0, j}], {m, 1, n}],
   {j, 1, d}
  ];
```

```
Table[Fd[n], {d, 0, 5}]
```

$$\left\{1,\ \frac{1}{4}(-1+n)\,n,\ \frac{1}{288}n\left(-10+15n-14n^2+9n^3\right),\ \frac{(-1+n)^2 n^2\left(10+n+3n^2\right)}{1152},\ \frac{1}{4147200}n\left(1488+2500n-7980n^2+10555n^3-9888n^4+3550n^5-900n^6+675n^7\right),\ \frac{1}{16588800}(-1+n)^2 n^2\left(-1488-3988n+992n^2-1223n^3+1105n^4+195n^5+135n^6\right)\right\}$$

```
IQd_[x_] := Sum[Fd[n] x^n / n!, {n, 0, ∞}]
```

```
Table[IQd[x], {d, 0, 5}]
```

$$\left\{e^x,\ \frac{e^x x^2}{4},\ \frac{1}{288}e^x x^2\left(36+40x+9x^2\right),\ \frac{e^x x^2\left(48+192x+156x^2+40x^3+3x^4\right)}{1152},\ \frac{1}{4147200}e^x x^2\left(43200+508800x+975600x^2+626112x^3+164200x^4+18000x^5+675x^6\right),\ \frac{1}{16588800}e^x x^2\left(34560+1152000x+4564800x^2+5719680x^3+3045840x^4+779712x^5+99400x^6+6000x^7+135x^8\right)\right\}$$

```
Table[Qd[x], {d, 0, 5}]
```

$$\left\{e^x,\ -\frac{1}{4}e^x x^2,\ \frac{1}{288}e^x x^3(32+9x),\ -\frac{e^x x^2\left(-24+72x^2+32x^3+3x^4\right)}{1152},\ \frac{1}{4147200}e^x x^3\left(-115200-21600x+165888x^2+90400x^3+14400x^4+675x^5\right),\ -\frac{1}{16588800}e^x x^2\left(34560-518400x^2-153600x^3+450000x^4+281088x^5+58000x^6+4800x^7+135x^8\right)\right\}$$