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 $\phi_0[x_] := e^x;$ 
 $\phi_k[x_] := \text{Module}[\{t\}, \text{Simplify}[\frac{1}{t^k} (e^t - \text{Normal@Series}[e^t, \{t, 0, k-1\}]) /. t \to x]]];$ 

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Table[ $\phi_k[x]$ , {k, 0, 3}]

$$\left\{ e^x, \frac{-1 + e^x}{x}, \frac{-1 + e^x - x}{x^2}, -\frac{2 - 2e^x + 2x + x^2}{2x^3} \right\}$$

ser = Series[

$$\frac{\mu t}{\hbar} (A^{-1} \phi_1[\mu (q-1)x] - 1) - \frac{\mu}{\hbar} (A \phi_1[\mu (q^{-1}-1)x] - 1) /. \{A \to e^{-\hbar \beta a}, q \to e^{\hbar \alpha \beta}\}, \{\beta, 0, 5\}]$$

$$\frac{1}{2} (2 a \mu + 2 a t \mu + x \alpha \mu^2 + t x \alpha \mu^2) \beta +$$

$$\left( -\frac{1}{12} \mu (6 a^2 + 6 a x \alpha \mu + 3 x \alpha^2 \mu + 2 x^2 \alpha^2 \mu^2) \hbar + \frac{1}{12} t \mu (6 a^2 + 6 a x \alpha \mu + 3 x \alpha^2 \mu + 2 x^2 \alpha^2 \mu^2) \hbar \right) \beta^2 +$$

$$\left( \frac{1}{24} \mu (4 a^3 + 6 a^2 x \alpha \mu + 6 a x \alpha^2 \mu + 2 x \alpha^3 \mu + 4 a x^2 \alpha^2 \mu^2 + 4 x^2 \alpha^3 \mu^2 + x^3 \alpha^3 \mu^3) \hbar^2 +$$

$$\frac{1}{24} t \mu (4 a^3 + 6 a^2 x \alpha \mu + 6 a x \alpha^2 \mu + 2 x \alpha^3 \mu + 4 a x^2 \alpha^2 \mu^2 + 4 x^2 \alpha^3 \mu^2 + x^3 \alpha^3 \mu^3) \hbar^2 \right) \beta^3 +$$

$$\left( -\frac{1}{720} \mu (30 a^4 + 60 a^3 x \alpha \mu + 90 a^2 x \alpha^2 \mu + 60 a x \alpha^3 \mu + 15 x \alpha^4 \mu + 60 a^2 x^2 \alpha^2 \mu^2 +$$

$$120 a x^2 \alpha^3 \mu^2 + 70 x^2 \alpha^4 \mu^2 + 30 a x^3 \alpha^3 \mu^3 + 45 x^3 \alpha^4 \mu^3 + 6 x^4 \alpha^4 \mu^4) \hbar^3 +$$

$$\frac{1}{720} t \mu (30 a^4 + 60 a^3 x \alpha \mu + 90 a^2 x \alpha^2 \mu + 60 a x \alpha^3 \mu + 15 x \alpha^4 \mu + 60 a^2 x^2 \alpha^2 \mu^2 +$$

$$120 a x^2 \alpha^3 \mu^2 + 70 x^2 \alpha^4 \mu^2 + 30 a x^3 \alpha^3 \mu^3 + 45 x^3 \alpha^4 \mu^3 + 6 x^4 \alpha^4 \mu^4) \hbar^3 \right) \beta^4 +$$

$$\left( \frac{1}{1440} \mu (12 a^5 + 30 a^4 x \alpha \mu + 60 a^3 x \alpha^2 \mu + 60 a^2 x \alpha^3 \mu + 30 a x \alpha^4 \mu + 6 x \alpha^5 \mu +$$

$$40 a^3 x^2 \alpha^2 \mu^2 + 120 a^2 x^2 \alpha^3 \mu^2 + 140 a x^2 \alpha^4 \mu^2 + 60 x^2 \alpha^5 \mu^2 + 30 a^2 x^3 \alpha^3 \mu^3 +$$

$$90 a x^3 \alpha^4 \mu^3 + 75 x^3 \alpha^5 \mu^3 + 12 a x^4 \alpha^4 \mu^4 + 24 x^4 \alpha^5 \mu^4 + 2 x^5 \alpha^5 \mu^5) \hbar^4 +$$

$$\frac{1}{1440} t \mu (12 a^5 + 30 a^4 x \alpha \mu + 60 a^3 x \alpha^2 \mu + 60 a^2 x \alpha^3 \mu + 30 a x \alpha^4 \mu + 6 x \alpha^5 \mu +$$

$$40 a^3 x^2 \alpha^2 \mu^2 + 120 a^2 x^2 \alpha^3 \mu^2 + 140 a x^2 \alpha^4 \mu^2 + 60 x^2 \alpha^5 \mu^2 + 30 a^2 x^3 \alpha^3 \mu^3 +$$

$$90 a x^3 \alpha^4 \mu^3 + 75 x^3 \alpha^5 \mu^3 + 12 a x^4 \alpha^4 \mu^4 + 24 x^4 \alpha^5 \mu^4 + 2 x^5 \alpha^5 \mu^5) \hbar^4 \right) \beta^5 + O[\beta]^6$$

**Collect[Normal@ser,  $\beta$ , Factor]**

$$\begin{aligned} & \frac{1}{2} (1+t) \beta \mu (2a + x \alpha \mu) + \frac{1}{12} (-1+t) \beta^2 \mu (6a^2 + 6a x \alpha \mu + 3x \alpha^2 \mu + 2x^2 \alpha^2 \mu^2) \hbar + \\ & \frac{1}{24} (1+t) \beta^3 \mu (4a^3 + 6a^2 x \alpha \mu + 6a x \alpha^2 \mu + 2x \alpha^3 \mu + 4a x^2 \alpha^2 \mu^2 + 4x^2 \alpha^3 \mu^2 + x^3 \alpha^3 \mu^3) \hbar^2 + \\ & \frac{1}{720} (-1+t) \beta^4 \mu (30a^4 + 60a^3 x \alpha \mu + 90a^2 x \alpha^2 \mu + 60a x \alpha^3 \mu + 15x \alpha^4 \mu + \\ & 60a^2 x^2 \alpha^2 \mu^2 + 120a x^2 \alpha^3 \mu^2 + 70x^2 \alpha^4 \mu^2 + 30a x^3 \alpha^3 \mu^3 + 45x^3 \alpha^4 \mu^3 + 6x^4 \alpha^4 \mu^4) \hbar^3 + \\ & \frac{1}{1440} (1+t) \beta^5 \mu (12a^5 + 30a^4 x \alpha \mu + 60a^3 x \alpha^2 \mu + 60a^2 x \alpha^3 \mu + 30a x \alpha^4 \mu + 6x \alpha^5 \mu + \\ & 40a^3 x^2 \alpha^2 \mu^2 + 120a^2 x^2 \alpha^3 \mu^2 + 140a x^2 \alpha^4 \mu^2 + 60x^2 \alpha^5 \mu^2 + 30a^2 x^3 \alpha^3 \mu^3 + \\ & 90a x^3 \alpha^4 \mu^3 + 75x^3 \alpha^5 \mu^3 + 12a x^4 \alpha^4 \mu^4 + 24x^4 \alpha^5 \mu^4 + 2x^5 \alpha^5 \mu^5) \hbar^4 \end{aligned}$$

$$\begin{aligned} \text{GS1} = & -\mu \left( -1 + \frac{e^{-a\beta\hbar} (-1 + e^{(-1+e^{-a\beta\hbar}) x \mu \hbar})}{(-1 + e^{-\alpha\beta\hbar}) x \mu \hbar} \right) + \text{T} \mu \left( -1 + \frac{e^{a\beta\hbar} (-1 + e^{(-1+e^{a\beta\hbar}) x \mu \hbar})}{(-1 + e^{\alpha\beta\hbar}) x \mu \hbar} \right) \\ & -\mu \left( -1 + \frac{e^{-a\beta\hbar} (-1 + e^{(-1+e^{-a\beta\hbar}) x \mu \hbar})}{(-1 + e^{-\alpha\beta\hbar}) x \mu \hbar} \right) + \text{T} \mu \left( -1 + \frac{e^{a\beta\hbar} (-1 + e^{(-1+e^{a\beta\hbar}) x \mu \hbar})}{(-1 + e^{\alpha\beta\hbar}) x \mu \hbar} \right) \end{aligned}$$

**GS2 = Collect[Normal@Series[GS1, { $\beta$ , 0, 3}],  $\beta$ , Factor]**

$$\begin{aligned} & \frac{1}{2} (1+\text{T}) \beta \mu \hbar (2a + x \alpha \mu \hbar) + \frac{1}{12} (-1+\text{T}) \beta^2 \mu \hbar^2 (6a^2 + 6a x \alpha \mu \hbar + 3x \alpha^2 \mu \hbar + 2x^2 \alpha^2 \mu^2 \hbar^2) + \frac{1}{24} \\ & (1+\text{T}) \beta^3 \mu \hbar^3 (4a^3 + 6a^2 x \alpha \mu \hbar + 6a x \alpha^2 \mu \hbar + 2x \alpha^3 \mu \hbar + 4a x^2 \alpha^2 \mu^2 \hbar^2 + 4x^2 \alpha^3 \mu^2 \hbar^2 + x^3 \alpha^3 \mu^3 \hbar^3) \end{aligned}$$

**\$TD = 5; Series[GS1 - GS2, { $\hbar$ , 0, \$TD}]**

$$\begin{aligned} & \frac{1}{24} (-a^4 \beta^4 \mu + a^4 \text{T} \beta^4 \mu) \hbar^4 + \\ & \frac{1}{240} (2a^5 \beta^5 \mu + 2a^5 \text{T} \beta^5 \mu - 20a^3 x \alpha \beta^4 \mu^2 + 20a^3 \text{T} x \alpha \beta^4 \mu^2 - 30a^2 x \alpha^2 \beta^4 \mu^2 + \\ & 30a^2 \text{T} x \alpha^2 \beta^4 \mu^2 - 20a x \alpha^3 \beta^4 \mu^2 + 20a \text{T} x \alpha^3 \beta^4 \mu^2 - 5x \alpha^4 \beta^4 \mu^2 + 5\text{T} x \alpha^4 \beta^4 \mu^2) \hbar^5 + \text{O}[\hbar]^6 \end{aligned}$$

**Series[ $e^x - 1$ , { $x$ , 0, 3}]**

$$x + \frac{x^2}{2} + \frac{x^3}{6} + \text{O}[x]^4$$

**Series[ $x(e^x - 1)$ , { $x$ , 0, 3}]**

$$x^2 + \frac{x^3}{2} + \text{O}[x]^4$$

**Series[ $x^7$ , { $x$ , 0, 4}]**

$$x^7 + \text{O}[x]^8$$