Pensieve header: Figuring out the double of the 2D pencil; Now with the opposite co-product! Older versions under “MetaDouble-*” and “Doubling-*”.

Issue: Improve DUForm.

```
$TD = 3; ℏ /: ℏ^d_. /; d > $TD := 0;
```

The 2D Lie BiAlgebra Pencil

The double is of the form $H^{*\mathrm{cop}} \otimes H$.
CHECK THIS! We hope to stick to $A = e^{-\hbar\beta a}$ and to $B = e^{-\hbar\alpha b}$, where $[x, a] = -\alpha x$ and $[b, y] = -\beta y$.
Also, $\Delta_{12}(a, A, x, b, B, y) = (a_1 + a_2,\ A_1 A_2,\ x_1 + A_1 x_2,\ b_1 + b_2,\ B_1 B_2,\ y_1 B_2 + y_2)$.
Also, (a, x) and $\hbar(y, b)$ are dual bases.

```
q = e^(ℏ α β);
```

```
AlgebraAtom = a | A[_] | x | b | B[_] | y;
$PBWRule = {y → 1, B[_] → 2, b → 3, A[_] → 4, a → 5, x → 6};
```

```
B[a, x] = α x; B[x, A[n_]] = (e^(n ℏ α β) - 1) U[A[n], x]; B[a, A[_]] = 0;
S[a] = -a; S[A[n_]] := A[-n]; S[x] = - U[A[-1], x];
Si[a] = -a; Si[A[n_]] := A[-n]; Si[x] = - U[x, A[-1]];
Δ[a] = U₁[a] U₂[] + U₁[] U₂[a]; Δ[A[n_]] := U₁[A[n]] U₂[A[n]];
Δ[x] = U₁[x] U₂[] + U₁[A[1]] U₂[x];
B[y, b] = β y; B[B[n_], y] = (e^(n ℏ α β) - 1) U[y, B[n]]; B[b, B[_]] = 0;
S[b] = -b; S[B[n_]] := B[-n]; S[y] = - U[y, B[-1]];
Si[b] = -b; Si[B[n_]] := B[-n]; Si[y] = - U[B[-1], y];
Δ[b] = U₁[b] U₂[] + U₁[] U₂[b]; Δ[B[n_]] := U₁[B[n]] U₂[B[n]];
Δ[y] = U₁[y] U₂[B[1]] + U₁[] U₂[y];
(* This extra line is annoying *)
```

```
ExpandAB[ℰ_] := Expand@Normal@Series[ℰ //. {
    c_. U_i_[λ___, A[n_], ρ___] :>
      Expand[c Sum[((-1)^d ℏ^d β^d n^d)/d! U_i[λ, Sequence @@ Table[a, {d}], ρ], {d, 0, $TD}]],
    c_. U_i_[λ___, B[n_], ρ___] :> Expand[
      c Sum[((-1)^d ℏ^d α^d n^d)/d! U_i[λ, Sequence @@ Table[b, {d}], ρ], {d, 0, $TD}]]
   },
  {ℏ,
   0,
   $TD}]
```

UEA with provisional modification

This section is based on penseive://Projects/UEA/.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_{i_}[] := U_i[]; UU_{i_}[1] := U_i[];
UU_{i_}[x_[n_]^{p_}] := U_i[x[n p]];
UU_{i_}[x_^{p_}] := UU_i @@ Table[x, {p}];
UU_{i_}[ℰ_] := ℰ /. {
     U[xs__] :→ UU_i[xs],
     x : AlgebraAtom :→ U_i[x]
    };
UU_{i_}[x_, xs__] := UU_{t1}[x] UU_{t2}[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_{s__}[0] = 0;
m_{s__}[x_Plus] := m_s /@ x;
m_{i_→j_}[ℰ_] := ℰ /. U_i → U_j;
```

```
m_{i_,j_→k_}[c_. U_{i_}[x___] U_{j_}[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_{i_}[] U_{j_}[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_{i_}[xx___, x_[n1_]] U_{j_}[x_[n2_], yy___]] :=
  USimp[c If[TrueQ[n1 + n2 == 0], U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_{i,j→k}];
m_{i_,j_→k_}[c_. U_{i_}[xx___, x_] U_{j_}[y_, yy___]] := If[x ≤ y,
    c U_k[xx, x, y, yy],
    ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
      c // USimp
   ];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, U_{i_}[___] :→ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
   {Sx = Supp[x], Sy = Supp[y], is, σ, z},
   If[MatchQ[Sx ⋃ Sy, {_Integer ...}] && Min[Sx ⋃ Sy] < 0,
    is = Abs[Sx] ⋂ Abs[Sy];
    z = x; Do[z = m_{-i→-σ@i}[m_{i→σ@i}[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_{σ@i,i→i}[z], {i, is}];
    z,
    (* else *) is = Sx ⋂ Sy;
    z = x; Do[z = m_{i→σ@i}[z], {i, is}];
    z = USimp[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}];
    z
   ]
  ];
UB[x_, y_] := USimp[x ** y - y ** x];
```

```
𝕆[func_] := Normal@Series[func, {ℏ, 0, $TD}];
𝕆[specs_, func_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] :→ Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[𝕆[func], vars] /. (ps_ → c_) :→ c (
        specs /. MapThread[((#1 → _) :→ #3^#2) &, {vars, ps, elems}] /. U_i_ :→ UU_i
       )]
 ]
```

The 2D Lie BiAlgebra Pencil, Testing

```
𝕆[U₁[a → a], ⅇ^(-ℏ β a)]
```

$U_1[\,] - \beta\hbar\, U_1[a] + \frac{1}{2}\beta^2\hbar^2\, U_1[a, a] - \frac{1}{6}\beta^3\hbar^3\, U_1[a, a, a]$

```
USimp@With[{An = 𝕆[U₁[a → a], ⅇ^(-n ℏ β a)]}, UB[U₁[x], An] - 𝕆[ⅇ^(n ℏ α β) - 1] An ** U₁[x]]
```

0

```
$TD = 6;
USimp@With[{Bn = 𝕆[U₁[b → b], ⅇ^(-n ℏ α b)]}, UB[Bn, U₁[y]] - 𝕆[ⅇ^(n ℏ α β) - 1] U₁[y] ** Bn]
```

0

```
z = U₁[a, A[2], x, x, x] U₂[a, a, x] U₃[a, a, A[-3], x];
(z // m_{1,2→1} // m_{1,3→1}) - (z // m_{2,3→2} // m_{1,2→1})
```

0

```
z = U₁[y, y, y, b, B[2]] U₂[y, b, b] U₃[y, b, b, B[-3]];
(z // m_{1,2→1} // m_{1,3→1}) - (z // m_{2,3→2} // m_{1,2→1})
```

0

The Co-Product and Co-Associativity

```
Δi_→j_,k_[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
      Ui[] → Uj[] Uk[],
      Ui[f_, fs___] ⧴
       (USimp[(Δ[f] /. {U1 → Uj, U2 → Uk}) Δi→tj,tk[Ui[fs]]] // mj,tj→j // mk,tk→k)
     }];
Δi_→j_,k_,l__[ℰ_] := ℰ // Δi→j,k // Δk→k,l
```

```
Δ1→1,2[U1[#]] & /@ {a, A[7], x, y, b, B[-3]}
```

{U1[a] U2[] + U1[] U2[a], U1[A[7]] U2[A[7]], U1[x] U2[] + U1[A[1]] U2[x],
 U1[] U2[y] + U1[y] U2[B[1]], U1[b] U2[] + U1[] U2[b], U1[B[-3]] U2[B[-3]]}

```
{lhs = U1[x] // Δ1→1,2 // Δ2→2,3, rhs = U1[x] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
```

{U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x],
 U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x], True}

```
U1[y] // Δ1→1,2
```

U1[] U2[y] + U1[y] U2[B[1]]

```
{lhs = U1[y] // Δ1→1,2 // Δ2→2,3, rhs = U1[y] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
```

{U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]],
 U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]], True}

```
z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
```

0

```
z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
```

0

The Antipode

```
Si_[ℰ_] := Module[{ti}, USimp[
    ℰ /. Ui[x_, xs___] ⧴ mti,i→i[Expand[UUi[S[x]] Sti[Uti[xs]]]]
   ]];
Sii_[ℰ_] := Module[{ti}, USimp[
    ℰ /. Ui[x_, xs___] ⧴ mti,i→i[Expand[UUi[Si[x]] Siti[Uti[xs]]]]
   ]];
```

```
{U1[x] // S1 // S1, U1[y] // S1 // S1}
```

$\{e^{\alpha\beta\hbar} U_1[x], e^{\alpha\beta\hbar} U_1[y]\}$

```
{U1[x] // S1 // Si1, U1[y] // S1 // Si1}
```

```
{U1[x], U1[y]}
```

```
{z = U1[]; (z // Δ1→1,2 // S1 // m1,2→1), z = U1[]; (z // Δ1→1,2 // S2 // m1,2→1)}
```

```
{U1[], U1[]}
```

```
z = U1[a, A[3], x, x];
{z // Δ1→1,2 // S2 // m1,2→1, z // Δ1→1,2 // S1 // m1,2→1}
```

```
{0, 0}
```

```
z = U1[y, y, y, b, B[-3]];
{z // Δ1→1,2 // S2 // m1,2→1, z // Δ1→1,2 // S1 // m1,2→1}
```

```
{0, 0}
```

```
{z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x]; (z // m1,2→1 // S1) - (z // S1 // S2 // m2,1→1),
 z = U1[y, y, y, b, B[2]] U2[y, b, b, b, B[6]]; (z // m1,2→1 // S1) - (z // S1 // S2 // m2,1→1)}
```

```
{0, 0}
```

```
{z = U1[a, A[2], x, x, x]; (z // S1 // Δ1→2,1) - (z // Δ1→1,2 // S1 // S2),
 z = U1[y, y, y, b, B[2]]; (z // S1 // Δ1→2,1) - (z // Δ1→1,2 // S1 // S2)
}
```

```
{0, 0}
```

The Pairing

```
P[U[], U[A[_]]] = P[U[B[_]], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
( P[U[b], U[a]] = ℏ^-1            P[U[b], U[A[n_]]] = -n β              P[U[b], U[x]] = 0
  P[U[B[n_]], U[a]] = -n α        P[U[B[n_]], U[A[m_]]] = e^(n m ℏ α β)  P[U[B[_]], U[x]] = 0
  P[U[y], U[a]] = 0               P[U[y], U[A[_]]] = 0                   P[U[y], U[x]] = ℏ^-1 );
```

```
P[U[], U[]] = 1;
Pi_,j_[ℰ_] := USimp[ℰ /. Ui[ys___] Uj[xs___] → P[U[ys], U[xs]]];
```

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[y_, ys__], U[x_]] := P[U[y, ys], U[x]] =
   Module[{i, j, k, l}, USimp[Ui[y] UUj[ys] Δk→k,l[Uk[x]]] // Pi,k // Pj,l];
P[U[ys__], U[x_, xs__]] := P[U[ys], U[x, xs]] =
   Module[{i, j, k, l}, USimp[Δi→i,j[UUi[ys]] Uk[x] UUl[xs]] // Pi,k // Pj,l];
```

```
P[U[b], U[a, a]]
```

```
0
```

```
P[U[b, b, b, b], U[a, a, a, a]]
```

$$\frac{24}{\hbar^4}$$

```
z = U_i[a] U_j[x] U_k[y];
{m_{i,j→i}[z] - m_{j,i→i}[z], Δ_{k→k,l}[z] - Δ_{k→l,k}[z]}
```

$\{\alpha\, U_i[x]\, U_k[y],\ -U_i[a]\, U_j[x]\, U_k[y]\, U_l[] + U_i[a]\, U_j[x]\, U_k[]\, U_l[y] - U_i[a]\, U_j[x]\, U_k[B[1]]\, U_l[y] + U_i[a]\, U_j[x]\, U_k[y]\, U_l[B[1]]\}$

```
Table[z = U_i[xi] U_j[xj] U_k[yk];
 {(m_{i,j→i}[z] - m_{j,i→i}[z]) // P_{k,i}, (Δ_{k→k,l}[z] - Δ_{k→l,k}[z]) // P_{k,i} // P_{l,j}},
 {xi, {a, x}}, {xj, {a, x}}, {yk, {b, y}}]
```

$$\left\{\left\{\{\{0,0\},\{0,0\}\},\left\{\{0,0\},\left\{\frac{\alpha}{\hbar},\frac{\alpha}{\hbar}\right\}\right\}\right\},\left\{\left\{\{0,0\},\left\{-\frac{\alpha}{\hbar},-\frac{\alpha}{\hbar}\right\}\right\},\{\{0,0\},\{0,0\}\}\right\}\right\}$$

```
Table[z = U_i[xi] U_k[yk] U_l[yl];
 {(Δ_{i→i,j}[z] - Δ_{i→j,i}[z]) // P_{k,i} // P_{l,j}, (m_{k,l→k}[z] - m_{l,k→k}[z]) // P_{k,i}},
 {xi, {a, x}}, {yk, {b, y}}, {yl, {b, y}}]
```

$$\left\{\{\{\{0,0\},\{0,0\}\},\{\{0,0\},\{0,0\}\}\},\left\{\left\{\{0,0\},\left\{-\frac{\beta}{\hbar},-\frac{\beta}{\hbar}\right\}\right\},\left\{\left\{\frac{\beta}{\hbar},\frac{\beta}{\hbar}\right\},\{0,0\}\right\}\right\}\right\}$$

```
lhs = Factor@Table[ħ^n P[U @@ Table[y, {n}], U @@ Table[x, {n}]], {n, $TD = 7}]
```

$$\Big\{1,\ 1+e^{\alpha\beta\hbar},\ \left(1+e^{\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right),\ \left(1+e^{\alpha\beta\hbar}\right)^2\left(1+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^2\left(1+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^3\left(1+e^{2\alpha\beta\hbar}\right)\left(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)^2\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^3\left(1+e^{2\alpha\beta\hbar}\right)\left(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)^2$$
$$\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}+e^{5\alpha\beta\hbar}+e^{6\alpha\beta\hbar}\right)\Big\}$$

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^{ħ α β}], {n, $TD = 7}]
```

$$\Big\{1,\ 1+e^{\alpha\beta\hbar},\ \left(1+e^{\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right),\ \left(1+e^{\alpha\beta\hbar}\right)^2\left(1+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^2\left(1+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^3\left(1+e^{2\alpha\beta\hbar}\right)\left(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)^2\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right),$$
$$\left(1+e^{\alpha\beta\hbar}\right)^3\left(1+e^{2\alpha\beta\hbar}\right)\left(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}\right)^2$$
$$\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}\right)\left(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}+e^{5\alpha\beta\hbar}+e^{6\alpha\beta\hbar}\right)\Big\}$$

```
MapThread[Equal, {lhs, rhs}]
```

{True, True, True, True, True, True, True}

```
Table[P[z1, z2],
  {z1, {U[], U[b], U[y], U[b, b], U[y, b], U[y, y], U[b, b, b],
    U[y, b, b], U[y, y, b], U[y, y, y]}}, {z2, {U[], U[a], U[x], U[a, a],
    U[a, x], U[x, x], U[a, a, a], U[a, a, x], U[a, x, x], U[x, x, x]}}
 ] //
 MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\hbar} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\hbar} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\hbar^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\hbar^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^2}+\frac{e^{\alpha\beta\hbar}}{\hbar^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{\hbar^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\hbar^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^3}+\frac{e^{\alpha\beta\hbar}}{\hbar^3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^3}+\frac{2e^{\alpha\beta\hbar}}{\hbar^3}+\frac{2e^{2\alpha\beta\hbar}}{\hbar^3}+\frac{e^{3\alpha\beta\hbar}}{\hbar^3} \end{pmatrix}$$

Pairing axioms:

```
$TD = 5;
(U₁[a, x, x] S₋₁[U₋₁[y, y, b]] // Expand // P₋₁,₁) -
  (S₁[U₁[a, x, x]] U₋₁[y, y, b] // Expand // P₋₁,₁) // ExpandAB
```

0

```
((U₁[a, x, x] U₂[a, A[1], x] U₋₁[y, y, y, b, b, b] // m₁,₂→₁ // Expand) // P₋₁,₁) -
 ((U₁[a, x, x] U₂[a, A[1], x] U₋₁[y, y, y, b, b, b] // Δ₋₁->₋₁,₋₂ // Expand) // P₋₁,₁ // P₋₂,₂)
```

0

```
((U₁[a, x, x, x, x] U₋₂[y, B[1], b] U₋₁[y, y, y, b, b, b] // m₋₁,₋₂→₋₁ // Expand) // P₋₁,₁) -
 ((U₁[a, x, x, x, x] U₋₂[y, B[1], b] U₋₁[y, y, y, b, b, b] // Δ₁->₁,₂ // Expand) // P₋₁,₁ // P₋₂,₂)
```

0

The Double

```
DUForm[ℰ_] := ℰ //. Uᵢ_[fs___] Uⱼ_[gs___] /; i + j == 0 ∧ j > 0 :→ "DU"ⱼ[fs, gs];
DUᵢ_[a] := U₋ᵢ[] Uᵢ[a]; DUᵢ_[A[n_]] := U₋ᵢ[] Uᵢ[A[n]]; DUᵢ_[x] := U₋ᵢ[] Uᵢ[x];
DUᵢ_[y] := U₋ᵢ[y] Uᵢ[]; DUᵢ_[b] := U₋ᵢ[b] Uᵢ[]; DUᵢ_[B[n_]] := U₋ᵢ[B[n]] Uᵢ[];
DUᵢ_[] := U₋ᵢ[] Uᵢ[];
```

```
DU₁ /@ {a, A[1], x, y, b, B[-1]}
```

{U₋₁[] U₁[a], U₋₁[] U₁[A[1]], U₋₁[] U₁[x], U₋₁[y] U₁[], U₋₁[b] U₁[], U₋₁[B[-1]] U₁[]}

```
dm_{i_,j_→k_}[ℰ_] := Module[{t1, t2, t3, h1, h2, h3},
   ℰ // Δ_{i→h1,h2,h3} // Si_{h3} // Δ_{-j→t1,t2,t3} // P_{t1,h3} // P_{t3,h1} // m_{h2,j→k} // m_{-i,t2→-k}];
dΔ_{i_→j_,k_}[ℰ_] := ℰ // Δ_{i→j,k} // Δ_{-i→-k,-j}; (*using opposite comult on the dual!*)
dS_{i_}[ℰ_] := Module[{h}, (ℰ // m_{i→h} // S_h // Si_{-i}) U_{-h}[] U_i[] // Expand // dm_{h,i→i}];
```

```
Module[{bas},
   bas = Join[{DU_1[]}, DU_1 /@ {a, A[n], x, y, B[m], b}, {β DU_1[a] - α DU_1[b]}];
   Table[If[f ** g =!= g ** f ∨ f === DU_1[] ∨ g === DU_1[], f ** g, "="], {f, bas}, {g, bas}] //
    ReplacePart[{1, 1} → "**"]
  ] // DUForm // MatrixForm
```

**	$DU_1[a]$	$DU_1[A[n]]$	$DU_1[x]$	$DU_1[y]$
$DU_1[a]$	=	=	$DU_1[a, x]$	$-\alpha\, DU_1[y] + DU$
$DU_1[A[n]]$	=	=	$DU_1[A[n], x]$	$e^{n\alpha\beta\hbar}\, DU_1[y,$
$DU_1[x]$	$-\alpha\, DU_1[x] + DU_1[a, x]$	$e^{n\alpha\beta\hbar}\, DU_1[A[n], x]$	=	$\frac{DU_1[]}{\hbar} + e^{\alpha\beta\hbar}\, DU_1[y, x]$
$DU_1[y]$	$DU_1[y, a]$	$DU_1[y, A[n]]$	$DU_1[y, x]$	=
$DU_1[B[m]]$	=	=	$DU_1[B[m], x]$	$e^{m\alpha\beta\hbar}\, DU_1[y,$
$DU_1[b]$	=	=	$DU_1[b, x]$	$-\beta\, DU_1[y] + DU$
$\beta\, DU_1[a] - \alpha\, DU_1[b]$	=	=	=	=

Associativity

```
z = U_{-1}[B[-2]] U_1[x] U_{-2}[y] U_2[A[-1]] U_{-3}[b] U_3[];
(z // dm_{1,2→1} // dm_{1,3→1}) - (z // dm_{2,3→2} // dm_{1,2→1}) // DUForm
```

0

```
z = U_{-1}[] U_1[x] U_{-2}[] U_2[a] U_{-3}[y] U_3[];
(z // dm_{1,2→1} // dm_{1,3→1}) - (z // dm_{2,3→2} // dm_{1,2→1}) // DUForm
```

0

```
$TD = 3; z = U_{-1}[y, b] U_1[A[2], a, x] U_{-2}[y, y, b] U_2[a] U_{-3}[y, b] U_3[a, a, x, x];
(z // dm_{1,2→1} // dm_{1,3→1}) - (z // dm_{2,3→2} // dm_{1,2→1}) // ExpandAB
```

0

```
(# → DUForm[dS_1[DU_1[#]]]) & /@ {a, x, y, b}
```

$\{a \to -DU_1[a],\ x \to -DU_1[A[-1], x],\ y \to -e^{-\alpha\beta\hbar}\, DU_1[y, B[-1]],\ b \to -DU_1[b]\}$

dS is anti algebra morphism

```
(DU_1[x] DU_2[y] // dm_{1,2→1} // dS_1) - (DU_1[x] DU_2[y] // dS_1 // dS_2 // dm_{2,1→1}) // ExpandAB
```

0

Coassociativity

```
(lhs = DU_1[#] // dΔ_{1→1,2} // dΔ_{2→2,3}; rhs = DU_1[#] // dΔ_{1→1,3} // dΔ_{1→1,2}; lhs == rhs) & /@
 {a, x, y, b}
```

{True, True, True, True}

dΔ is algebra morphism

```
z = U₋₁[] U₁[x] U₋₂[y] U₂[];
(z // dm1,2→1 // dΔ1→1,2) - (z // dΔ2→3,4 // dΔ1→1,2 // dm1,3→1 // dm2,4→2) // DUForm
```

0

```
z = U₋₁[y, b, B[2]] U₁[x] U₋₂[y, b, B[-3]] U₂[a, x];
(z // dm1,2→1 // dΔ1→1,2) - (z // dΔ2→3,4 // dΔ1→1,2 // dm1,3→1 // dm2,4→2) // DUForm
```

0

```
z = U₋₁[y, b, B[2]] U₁[x] U₋₂[y, b, B[-3]] U₂[a, x];
(z // dm1,2→1 // dΔ1→1,2) - (z // dΔ2→3,4 // dΔ1→1,2 // dm1,3→1 // dm2,4→2)
```

0

dS is the convolution inverse of dm.

```
z = U₋₁[y, y, b, B[-3]] U₁[a, A[1], x, x];
{z // dΔ1→1,2 // dS2 // dm1,2→1, z // dΔ1→1,2 // dS1 // dm1,2→1}
```

{0, 0}

```
z = U₋₁[y, b, B[-3]] U₁[a, x];
{z // dΔ1→1,2 // dS2 // dm1,2→1, z // dΔ1→1,2 // dS1 // dm1,2→1}
```

{0, 0}

The R-Matrix

Quesne's formula:

$$\mathbb{e}_{q\_}[x\_] := \text{Exp}\left[\sum_{k=1}^{\$TD} \frac{(1-q)^k}{k\,(1-q^k)} x^k\right];$$

```
Table[Together@SeriesCoefficient[ℯρ[x], {x, 0, n}], {n, 0, $TD}]
```

$$\left\{1, 1, \frac{1}{1+\rho}, \frac{1}{(1+\rho)\,(1+\rho+\rho^2)}\right\}$$

$$R_{i\_,j\_} := \mathbb{O}\left[U_{-i}[y\to y, b\to b]\, U_i[]\, U_{-j}[]\, U_j[a\to a, x\to x],\ \mathbb{e}^{\hbar\, b\, a}\, \mathbb{e}_q[\hbar\, x\, y]\right]$$

```
$TD = 3; R1,2 // DUForm
```

$$DU_1[]\,DU_2[] + \hbar\,DU_1[b]\,DU_2[a] + \hbar\,DU_1[y]\,DU_2[x] + \frac{1}{2}\hbar^2\,DU_1[b, b]\,DU_2[a, a] + \hbar^2\,DU_1[y, b]\,DU_2[a, x] + \frac{1}{2}\hbar^2\,DU_1[y, y]\,DU_2[x, x] - \frac{1}{4}\alpha\,\beta\,\hbar^3\,DU_1[y, y]\,DU_2[x, x] + \frac{1}{6}\hbar^3\,DU_1[b, b, b]\,DU_2[a, a, a] + \frac{1}{2}\hbar^3\,DU_1[y, b, b]\,DU_2[a, a, x] + \frac{1}{2}\hbar^3\,DU_1[y, y, b]\,DU_2[a, x, x] + \frac{1}{6}\hbar^3\,DU_1[y, y, y]\,DU_2[x, x, x]$$

```
$TD = 5; {lhs = R1,3 // dΔ1→1,2, rhs = R2,3 ** R1,3, lhs - rhs} // Last // ExpandAB // DUForm
```

0

```
$TD = 5; {lhs = R1,2 // dΔ2→2,3, rhs = R1,2 ** R1,3, lhs - rhs} // Last // ExpandAB // DUForm
```

0

```
$TD = 5; Table[
 {lhs = R1,2 ** dΔ1→2,1[f], rhs = dΔ1→1,2[f] ** R1,2, lhs - rhs} // Last // ExpandAB // DUForm,
 {f, {U-1[] U1[a], U-1[] U1[x], U-1[b] U1[], U-1[y] U1[]}}]
```

{0, 0, 0, 0}

```
$TD = 5;
{lhs = ExpandAB[R1,2 ** R1,3 ** R2,3],
  rhs = ExpandAB[R2,3 ** R1,3 ** R1,2], Coefficient[lhs - rhs, ℏ^$TD]} // Last
```

0

```
$TD = 5; dS2[R1,2] ** R1,2 // ExpandAB // DUForm
```

$DU_1[\,]\, DU_2[\,]$

Cuaps

Drinfeld element *u*.

```
ui_ := R1,2 // dS1 // dm2,1→i
```

```
$TD = 2; u1 // ExpandAB // DUForm
```

$$DU_1[\,] - \beta\hbar\, DU_1[a] - \alpha\hbar\, DU_1[b] + \frac{1}{2}\beta^2\hbar^2\, DU_1[a,a] - \hbar\, DU_1[b,a] + \alpha\beta\hbar^2\, DU_1[b,a] + \frac{1}{2}\alpha^2\hbar^2\, DU_1[b,b] - \hbar\, DU_1[y,x] - \alpha\beta\hbar^2\, DU_1[y,x] + \beta\hbar^2\, DU_1[b,a,a] + \alpha\hbar^2\, DU_1[b,b,a] + \frac{1}{2}\hbar^2\, DU_1[b,b,a,a] + \hbar^2\, DU_1[y,b,a,x] + \frac{1}{2}\hbar^2\, DU_1[y,y,x,x]$$

Conjugation by the Drinfeld element implements the square of the antipode.

```
$TD = 3; Table[DU1[f] ** u1 == u1 ** dS1[dS1[DU1[f]]] // ExpandAB, {f, {a, x, b, y}}]
```

{True, True, True, True}

Inverse of the Drinfeld element

```
uii_ := R1,2 // dS2 // dS2 // dm2,1→i
```

```
$TD = 3; ui1 ** u1 // ExpandAB
```

$U_{-1}[\,]\, U_1[\,]$

Drinfeld commutes with its antipode and the product is central

```
$TD = 3; u1 ** dS1[u1] - dS1[u1] ** u1 // ExpandAB
```

0

```
$TD = 3;
(u₁ ** dS₁[u₁] ** DU₁[#] - DU₁[#] ** u₁ ** dS₁[u₁]) & /@ {a, x, y, b} // ExpandAB
```

```
{0, 0, 0, 0}
```

Multiplying the inverse and antipode of Drinfeld

```
$TD = 3; ui₁ ** dS₁[u₁] - U₋₁[B[-1]] U₁[A[-1]] // ExpandAB // DUForm
```

```
0
```

This implies that
dS[u] = $u\,B^{-1}\,A^{-1}$. In other words, u dS[u] =
$u^2\,B^{-1}\,A^{-1}$ so we can take the square root and call it v. This is the ribbon element.

```
v_i_ := (𝕆[U₋ᵢ[b → b] Uᵢ[a → a], Exp[ħ/2 (α b + β a)]]) ** uᵢ
```

```
$TD = 3; Timing[u₁ ** dS₁[u₁] - v₁ ** v₁ // ExpandAB] (*takes long!*)
```

```
{725.141, 0}
```

v is supposed to be the Reidemeister 1 curl (with the appropriate cuaps/spinners added!). Note how it too is central.

```
$TD = 3; (v₁ ** DU₁[#] - DU₁[#] ** v₁) & /@ {a, x, y, b} // ExpandAB
```

```
{0, 0, 0, 0}
```

The rule for the cuaps is to add to a right-moving cup uv^{-1} and to a right – moving cap vu^{-1}. Note how $vu^{-1} = B^{-1/2}\,A^{-1/2}$

```
$TD = 3; v₁ ** ui₁ - 𝕆[U₋₁[b → b] U₁[a → a], Exp[ħ/2 (α b + β a)]] // ExpandAB
```

```
0
```

```
cap_i_ := 𝕆[U₋ᵢ[b → b] Uᵢ[a → a], Exp[ħ/2 (α b + β a)]]
cup_i_ := 𝕆[U₋ᵢ[b → b] Uᵢ[a → a], Exp[-ħ/2 (α b + β a)]]
```

Now let’s do R2 and oppositely oriented Reidemeister 2:

```
$TD = 3; dS₂[R₁,₂] R₃,₄ // dm₁,₃→₁ // dm₂,₄→₂ // ExpandAB // DUForm
```

```
DU₁[] DU₂[]
```

```
$TD = 3;
dS₂[R₁,₂] R₃,₄ cup₅ cap₆ // dm₁,₃→₁ // dm₄,₅→₄ // dm₄,₂→₄ // dm₄,₆→₄ // ExpandAB // DUForm
```

```
DU₁[] DU₄[]
```

Rotate a crossing:

```
$TD = 3;
(cup₁ dS₅[R₂,₅] cap₃ cup₄ cap₆ // dm₁,₂→₁ // dm₁,₃→₁ // dm₄,₅->₄ // dm₄,₆→₄)  - dS₄[R₁,₄] // ExpandAB
0
```

Reidemeister 1 curls, the two positives agree with the two negatives and they are inverses and agree with v, v^{-1}.

```
$TD = 2; {
  (R₁,₂ cap₃ // dm₁,₃→₁ // dm₁,₂→₁) - (R₁,₂ cup₃ // dm₂,₃→₂ // dm₂,₁→₁),
  (dS₂[R₁,₂] cup₃ // dm₁,₃→₁ // dm₁,₂→₁) - (dS₂[R₁,₂] cap₃ // dm₂,₃→₂ // dm₂,₁→₁),
  (dS₂[R₁,₂] cup₃ // dm₁,₃→₁ // dm₁,₂→₁) ** (R₁,₂ cap₃ // dm₁,₃→₁ // dm₁,₂→₁),
  (dS₂[R₁,₂] cup₃ // dm₁,₃→₁ // dm₁,₂→₁) - v₁} // ExpandAB // DUForm
{0, 0, DU₁[], 0}
```

The Central Element

```
c₁ = α DU₁[b] - β DU₁[a];
UB[c₁, #] & /@ {DU₁[a], DU₁[x], DU₁[y], DU₁[b]}
{0, 0, 0, 0}
```

$T = A^{-1} B$ is central, it allows us to eliminate B as B = TA.

```
DU₁[A[1]] ** DU₁[B[-1]] ** DU₁[x] - DU₁[x] ** DU₁[A[1]] ** DU₁[B[-1]]
0
```

Commuting Exponentials (other than $e^x\, e^y$)

Commuting e^a with e^x:

```
$TD = 5; 𝕆[U₁[x → x, a → a], e^(ℏ (μ x+ν a))] == 𝕆[U₁[a → a, x → x], e^(ℏ (μ e^(-ℏ α ν) x+ν a))]
True
```

Commuting e^b with e^y:

```
$TD = 5; 𝕆[U₁[b → b, y → y], e^(ℏ (μ y+ν b))] == 𝕆[U₁[y → y, b → b], e^(ℏ (μ e^(-ℏ β ν) y+ν b))]
True
```

The co-product of e^a:

```
$TD = 5; Δ₁→₁,₂[𝕆[U₁[a → a], e^(ℏ μ a)]] == 𝕆[U₁[a₁ → a] U₂[a₂ → a], e^(ℏ μ a₁) e^(ℏ μ a₂)]
True
```

The co-product of e_q^x:

```
$TD = 5 ; (𝕆[U₁[x → x], e_q[ℏ μ x]] // Δ₁→₁,₂ // ExpandAB) ==
 𝕆[U₁[a₁ → a, x₁ → x] U₂[x₂ → x], e_q[ℏ μ e^(-ℏ β a₁) x₂] e_q[ℏ μ x₁]]
True
```

The triple co-product of e_q^x:

```
$TD = 5; (𝕆[U₁[x → x], 𝕖q[ℏ μ x]] // Δ1→1,2,3 // ExpandAB) -
  𝕆[U₁[a₁ → a, x₁ → x] U₂[a₂ → a, x₂ → x] U₃[x₃ → x],
   𝕖q[ℏ μ 𝕖^(-ℏ β (a₁+a₂)) x₃] 𝕖q[ℏ μ 𝕖^(-ℏ β a₁) x₂] 𝕖q[ℏ μ x₁]]
```

0

The co-product of e_q^y:

```
$TD = 5 ; (𝕆[U₁[y → y], 𝕖q[ℏ ν y]] // Δ1→1,2 // ExpandAB) ==
  𝕆[U₁[y₁ → y] U₂[y₂ → y, b₂ → b], 𝕖q[ℏ ν y₂] 𝕖q[ℏ ν 𝕖^(-ℏ α b₂) y₁]]
```

True

The triple co-product of e_q^y:

```
$TD = 5 ; (𝕆[U₁[y → y], 𝕖q[ℏ ν y]] // Δ1→1,2,3 // ExpandAB) ==
  𝕆[U₁[y₁ → y] U₂[y₂ → y, b₂ → b] U₃[y₃ → y, b₃ → b],
   𝕖q[ℏ ν y₃] 𝕖q[ℏ ν 𝕖^(-ℏ α b₃) y₂] 𝕖q[ℏ ν 𝕖^(-ℏ α (b₂+b₃)) y₁]]
```

True

The inverse-antipode on e_q^x:

```
$TD = 8;
(𝕆[U₁[x → x], 𝕖q[ℏ μ x]] // Si₁ // ExpandAB) -
  (𝕆[U₁[a → a, x → x], Sum[((-ℏ μ)^k q^(-k (k+1)/2) 𝕖^(ℏ β k a) x^k) / QFactorial[k, q], {k, 0, $TD}]])
```

0

Pairing e_q^x with e_q^y:

```
$TD = 5;
$TD *= 2;
lhs = 𝕆[U₋₁[y → y], 𝕖q[ℏ ν y]] 𝕆[U₁[x → x], 𝕖q[ℏ μ x]] // Expand // P₋₁,₁ // ExpandAB;
($TD /= 2; 𝕖q[ℏ μ ν] - lhs // ExpandAB)
```

0

Computing $e^{\mu x} y e^{-\mu x}$.

```
TestRel[n_] := (U₋₁[] UU₁[x^n]) ** DU₁[y] - q^n U₋₁[y] UU₁[x^n] -
    (1 - q^n)/(1 - q) (DU₁[]/ℏ - q^(n-1) (U₋₁[B[1]] U₁[A[1]])/ℏ) ** (U₋₁[] UU₁[x^(n-1)]) // ExpandAB
$TD =
 4;
TestRel[5]
```

0

Following BBS:VanDerVeen-170620-212242, though with explicit $\hbar$ put into μ, ν.

```
ϕ0[x_] := ⅇ^x;
ϕk_[x_] := Module[{t}, Simplify[(ⅇ^t - Normal@Series[ⅇ^t, {t, 0, k - 1}]) / t^k /. t → x]];
```

```
Table[ϕk[x], {k, 0, 3}]
```

$$\left\{e^x,\ \frac{-1+e^x}{x},\ \frac{-1+e^x-x}{x^2},\ -\frac{2-2e^x+2x+x^2}{2x^3}\right\}$$

```
$TD = 5;
GS = T μ (A^-1 ϕ1[ℏ μ (q - 1) x] - 1) - μ (A ϕ1[ℏ μ (q^-1 - 1) x] - 1) /. {A -> ⅇ^(-ℏ β a)};
(G = 𝕆[U-1[] U1[x → x, a → a], GS]) // DUForm
{lhs = 𝕆[U-1[] U1[x → x], ⅇ^(ℏ μ x)] ** DU1[y] ** 𝕆[U-1[] U1[x → x], ⅇ^(-ℏ μ x)],
  rhs = Expand[DU1[y] + μ (T - 1) DU1[] + G],
  lhs - rhs /. T * ℰ_ :→ DU1[A[1]] ** DU1[B[1]] ** ℰ} // Last // ExpandAB // DUForm
```

$\beta\mu\hbar\,DU_1[a] + T\beta\mu\hbar\,DU_1[a] + \frac{1}{2}\alpha\beta\mu^2\hbar^2 DU_1[x] + \frac{1}{2}T\alpha\beta\mu^2\hbar^2 DU_1[x] +$
$\frac{1}{4}\alpha^2\beta^2\mu^2\hbar^3 DU_1[x] - \frac{1}{4}T\alpha^2\beta^2\mu^2\hbar^3 DU_1[x] + \frac{1}{12}\alpha^3\beta^3\mu^2\hbar^4 DU_1[x] + \frac{1}{12}T\alpha^3\beta^3\mu^2\hbar^4 DU_1[x] +$
$\frac{1}{48}\alpha^4\beta^4\mu^2\hbar^5 DU_1[x] - \frac{1}{48}T\alpha^4\beta^4\mu^2\hbar^5 DU_1[x] - \frac{1}{2}\beta^2\mu\hbar^2 DU_1[a, a] + \frac{1}{2}T\beta^2\mu\hbar^2 DU_1[a, a] -$
$\frac{1}{2}\alpha\beta^2\mu^2\hbar^3 DU_1[a, x] + \frac{1}{2}T\alpha\beta^2\mu^2\hbar^3 DU_1[a, x] - \frac{1}{4}\alpha^2\beta^3\mu^2\hbar^4 DU_1[a, x] -$
$\frac{1}{4}T\alpha^2\beta^3\mu^2\hbar^4 DU_1[a, x] - \frac{1}{12}\alpha^3\beta^4\mu^2\hbar^5 DU_1[a, x] + \frac{1}{12}T\alpha^3\beta^4\mu^2\hbar^5 DU_1[a, x] -$
$\frac{1}{6}\alpha^2\beta^2\mu^3\hbar^4 DU_1[x, x] + \frac{1}{6}T\alpha^2\beta^2\mu^3\hbar^4 DU_1[x, x] - \frac{1}{6}\alpha^3\beta^3\mu^3\hbar^5 DU_1[x, x] -$
$\frac{1}{6}T\alpha^3\beta^3\mu^3\hbar^5 DU_1[x, x] + \frac{1}{6}\beta^3\mu\hbar^3 DU_1[a, a, a] + \frac{1}{6}T\beta^3\mu\hbar^3 DU_1[a, a, a] +$
$\frac{1}{4}\alpha\beta^3\mu^2\hbar^4 DU_1[a, a, x] + \frac{1}{4}T\alpha\beta^3\mu^2\hbar^4 DU_1[a, a, x] + \frac{1}{8}\alpha^2\beta^4\mu^2\hbar^5 DU_1[a, a, x] -$
$\frac{1}{8}T\alpha^2\beta^4\mu^2\hbar^5 DU_1[a, a, x] + \frac{1}{6}\alpha^2\beta^3\mu^3\hbar^5 DU_1[a, x, x] + \frac{1}{6}T\alpha^2\beta^3\mu^3\hbar^5 DU_1[a, x, x] -$
$\frac{1}{24}\beta^4\mu\hbar^4 DU_1[a, a, a, a] + \frac{1}{24}T\beta^4\mu\hbar^4 DU_1[a, a, a, a] - \frac{1}{12}\alpha\beta^4\mu^2\hbar^5 DU_1[a, a, a, x] +$
$\frac{1}{12}T\alpha\beta^4\mu^2\hbar^5 DU_1[a, a, a, x] + \frac{1}{120}\beta^5\mu\hbar^5 DU_1[a, a, a, a, a] + \frac{1}{120}T\beta^5\mu\hbar^5 DU_1[a, a, a, a, a]$

$$
\begin{aligned}
&2\,\alpha\,\mu\,\hbar\,\mathrm{DU}_1[b] - 2\,\alpha\,\beta\,\mu^2\,\hbar^2\,\mathrm{DU}_1[x] - \alpha^2\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[x] - \frac{5}{6}\,\alpha^3\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[x] - \frac{1}{3}\,\alpha^4\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[x] + \\
&2\,\alpha\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[a, x] + \frac{3}{2}\,\alpha^2\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, x] + \frac{5}{6}\,\alpha^3\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, x] - \alpha\,\beta\,\mu\,\hbar^2\,\mathrm{DU}_1[b, a] - \\
&\alpha^2\,\mu\,\hbar^2\,\mathrm{DU}_1[b, b] + 2\,\alpha^2\,\beta\,\mu^2\,\hbar^3\,\mathrm{DU}_1[b, x] + \alpha^3\,\beta^2\,\mu^2\,\hbar^4\,\mathrm{DU}_1[b, x] + \frac{5}{6}\,\alpha^4\,\beta^3\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, x] - \\
&\alpha^2\,\beta^2\,\mu^3\,\hbar^4\,\mathrm{DU}_1[x, x] - \frac{3}{2}\,\alpha^3\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[x, x] + \alpha\,\beta\,\mu\,\hbar^2\,\mathrm{DU}_1[y, x] + \frac{1}{2}\,\alpha^2\,\beta^2\,\mu\,\hbar^3\,\mathrm{DU}_1[y, x] + \\
&\frac{1}{6}\,\alpha^3\,\beta^3\,\mu\,\hbar^4\,\mathrm{DU}_1[y, x] + \frac{1}{24}\,\alpha^4\,\beta^4\,\mu\,\hbar^5\,\mathrm{DU}_1[y, x] - \alpha\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, a, x] - \frac{3}{4}\,\alpha^2\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, x] + \\
&\alpha^2\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[a, x, x] + \frac{1}{2}\,\alpha\,\beta^2\,\mu\,\hbar^3\,\mathrm{DU}_1[b, a, a] - \frac{3}{2}\,\alpha^2\,\beta^2\,\mu^2\,\hbar^4\,\mathrm{DU}_1[b, a, x] - \\
&\frac{5}{4}\,\alpha^3\,\beta^3\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, a, x] + \frac{1}{2}\,\alpha^2\,\beta\,\mu\,\hbar^3\,\mathrm{DU}_1[b, b, a] + \frac{1}{3}\,\alpha^3\,\mu\,\hbar^3\,\mathrm{DU}_1[b, b, b] - \\
&\alpha^3\,\beta\,\mu^2\,\hbar^4\,\mathrm{DU}_1[b, b, x] - \frac{1}{2}\,\alpha^4\,\beta^2\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, b, x] + \frac{4}{3}\,\alpha^3\,\beta^2\,\mu^3\,\hbar^5\,\mathrm{DU}_1[b, x, x] + \\
&\frac{1}{2}\,\alpha^2\,\beta^2\,\mu^2\,\hbar^4\,\mathrm{DU}_1[y, x, x] + \frac{1}{2}\,\alpha^3\,\beta^3\,\mu^2\,\hbar^5\,\mathrm{DU}_1[y, x, x] + \frac{1}{3}\,\alpha\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, a, x] - \\
&\frac{1}{6}\,\alpha\,\beta^3\,\mu\,\hbar^4\,\mathrm{DU}_1[b, a, a, a] + \frac{3}{4}\,\alpha^2\,\beta^3\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, a, a, x] - \frac{1}{4}\,\alpha^2\,\beta^2\,\mu\,\hbar^4\,\mathrm{DU}_1[b, b, a, a] + \\
&\frac{3}{4}\,\alpha^3\,\beta^2\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, b, a, x] - \frac{1}{6}\,\alpha^3\,\beta\,\mu\,\hbar^4\,\mathrm{DU}_1[b, b, b, a] - \frac{1}{12}\,\alpha^4\,\mu\,\hbar^4\,\mathrm{DU}_1[b, b, b, b] + \\
&\frac{1}{3}\,\alpha^4\,\beta\,\mu^2\,\hbar^5\,\mathrm{DU}_1[b, b, b, x] + \frac{1}{24}\,\alpha\,\beta^4\,\mu\,\hbar^5\,\mathrm{DU}_1[b, a, a, a, a] + \frac{1}{12}\,\alpha^2\,\beta^3\,\mu\,\hbar^5\,\mathrm{DU}_1[b, b, a, a, a] + \\
&\frac{1}{12}\,\alpha^3\,\beta^2\,\mu\,\hbar^5\,\mathrm{DU}_1[b, b, b, a, a] + \frac{1}{24}\,\alpha^4\,\beta\,\mu\,\hbar^5\,\mathrm{DU}_1[b, b, b, b, a] + \frac{1}{60}\,\alpha^5\,\mu\,\hbar^5\,\mathrm{DU}_1[b, b, b, b, b]
\end{aligned}
$$

```
K = 3; $TD = 5;
GS1 = μ T (A^-1 ϕ1[ℏ μ (q - 1) x] - 1) - μ (A ϕ1[ℏ μ (q^-1 - 1) x] - 1) /. {A → e^(-ℏ β a)};
(G1 = 𝕆[U-1[] U1[x → x, a → a], GS1]) // DUForm
Collect[ExpandAB[
   Expand[DU1[y] + (T - 1) μ DU1[] + G1 - 𝕆[U-1[] U1[x → x], e^(ℏ μ x)] ** DU1[y] **
       𝕆[U-1[] U1[x → x], e^(-ℏ μ x)]] /. T * ℰ_ :> ℰ ** DU1[A[1]] ** DU1[B[1]]
  ],
 β]
```

$$\beta\,\mu\,\hbar\,\mathrm{DU}_1[a] + T\,\beta\,\mu\,\hbar\,\mathrm{DU}_1[a] + \frac{1}{2}\alpha\,\beta\,\mu^2\,\hbar^2\,\mathrm{DU}_1[x] + \frac{1}{2}T\,\alpha\,\beta\,\mu^2\,\hbar^2\,\mathrm{DU}_1[x] +$$
$$\frac{1}{4}\alpha^2\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[x] - \frac{1}{4}T\,\alpha^2\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[x] + \frac{1}{12}\alpha^3\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[x] + \frac{1}{12}T\,\alpha^3\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[x] +$$
$$\frac{1}{48}\alpha^4\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[x] - \frac{1}{48}T\,\alpha^4\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[x] - \frac{1}{2}\beta^2\,\mu\,\hbar^2\,\mathrm{DU}_1[a, a] + \frac{1}{2}T\,\beta^2\,\mu\,\hbar^2\,\mathrm{DU}_1[a, a] -$$
$$\frac{1}{2}\alpha\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[a, x] + \frac{1}{2}T\,\alpha\,\beta^2\,\mu^2\,\hbar^3\,\mathrm{DU}_1[a, x] - \frac{1}{4}\alpha^2\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, x] -$$
$$\frac{1}{4}T\,\alpha^2\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, x] - \frac{1}{12}\alpha^3\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, x] + \frac{1}{12}T\,\alpha^3\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, x] -$$
$$\frac{1}{6}\alpha^2\,\beta^2\,\mu^3\,\hbar^4\,\mathrm{DU}_1[x, x] + \frac{1}{6}T\,\alpha^2\,\beta^2\,\mu^3\,\hbar^4\,\mathrm{DU}_1[x, x] - \frac{1}{6}\alpha^3\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[x, x] -$$
$$\frac{1}{6}T\,\alpha^3\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[x, x] + \frac{1}{6}\beta^3\,\mu\,\hbar^3\,\mathrm{DU}_1[a, a, a] + \frac{1}{6}T\,\beta^3\,\mu\,\hbar^3\,\mathrm{DU}_1[a, a, a] +$$
$$\frac{1}{4}\alpha\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, a, x] + \frac{1}{4}T\,\alpha\,\beta^3\,\mu^2\,\hbar^4\,\mathrm{DU}_1[a, a, x] + \frac{1}{8}\alpha^2\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, x] -$$
$$\frac{1}{8}T\,\alpha^2\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, x] + \frac{1}{6}\alpha^2\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[a, x, x] + \frac{1}{6}T\,\alpha^2\,\beta^3\,\mu^3\,\hbar^5\,\mathrm{DU}_1[a, x, x] -$$
$$\frac{1}{24}\beta^4\,\mu\,\hbar^4\,\mathrm{DU}_1[a, a, a, a] + \frac{1}{24}T\,\beta^4\,\mu\,\hbar^4\,\mathrm{DU}_1[a, a, a, a] - \frac{1}{12}\alpha\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, a, x] +$$
$$\frac{1}{12}T\,\alpha\,\beta^4\,\mu^2\,\hbar^5\,\mathrm{DU}_1[a, a, a, x] + \frac{1}{120}\beta^5\,\mu\,\hbar^5\,\mathrm{DU}_1[a, a, a, a, a] + \frac{1}{120}T\,\beta^5\,\mu\,\hbar^5\,\mathrm{DU}_1[a, a, a, a, a]$$

$$-2\,\alpha\,\mu\,\hbar\,U_{-1}[b]\,U_1[\,] + \alpha^2\,\mu\,\hbar^2\,U_{-1}[b,b]\,U_1[\,] - \frac{1}{3}\,\alpha^3\,\mu\,\hbar^3\,U_{-1}[b,b,b]\,U_1[\,] +$$
$$\frac{1}{12}\,\alpha^4\,\mu\,\hbar^4\,U_{-1}[b,b,b,b]\,U_1[\,] - \frac{1}{60}\,\alpha^5\,\mu\,\hbar^5\,U_{-1}[b,b,b,b,b]\,U_1[\,] +$$
$$\beta\,\Big(\alpha\,\mu\,\hbar^2\,U_{-1}[b]\,U_1[a] - \frac{1}{2}\,\alpha^2\,\mu\,\hbar^3\,U_{-1}[b,b]\,U_1[a] + \frac{1}{6}\,\alpha^3\,\mu\,\hbar^4\,U_{-1}[b,b,b]\,U_1[a] -$$
$$\frac{1}{24}\,\alpha^4\,\mu\,\hbar^5\,U_{-1}[b,b,b,b]\,U_1[a] + 2\,\alpha\,\mu^2\,\hbar^2\,U_{-1}[\,]\,U_1[x] - 2\,\alpha^2\,\mu^2\,\hbar^3\,U_{-1}[b]\,U_1[x] -$$
$$\alpha\,\mu\,\hbar^2\,U_{-1}[y]\,U_1[x] + \alpha^3\,\mu^2\,\hbar^4\,U_{-1}[b,b]\,U_1[x] - \frac{1}{3}\,\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b,b,b]\,U_1[x]\Big) +$$
$$\beta^2\,\Big(2\,\alpha^2\,\mu^2\,\hbar^3\,U_{-1}[\,]\,U_1[x] - 2\,\alpha^3\,\mu^2\,\hbar^4\,U_{-1}[b]\,U_1[x] - \frac{1}{2}\,\alpha^2\,\mu\,\hbar^3\,U_{-1}[y]\,U_1[x] + \alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b,b]\,U_1[x] -$$
$$\frac{1}{2}\,\alpha\,\mu\,\hbar^3\,U_{-1}[b]\,U_1[a,a] + \frac{1}{4}\,\alpha^2\,\mu\,\hbar^4\,U_{-1}[b,b]\,U_1[a,a] - \frac{1}{12}\,\alpha^3\,\mu\,\hbar^5\,U_{-1}[b,b,b]\,U_1[a,a] -$$
$$2\,\alpha\,\mu^2\,\hbar^3\,U_{-1}[\,]\,U_1[a,x] + \frac{3}{2}\,\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[b]\,U_1[a,x] - \frac{3}{4}\,\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[b,b]\,U_1[a,x] +$$
$$\alpha^2\,\mu^3\,\hbar^4\,U_{-1}[\,]\,U_1[x,x] - \frac{4}{3}\,\alpha^3\,\mu^3\,\hbar^5\,U_{-1}[b]\,U_1[x,x] - \frac{1}{2}\,\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[y]\,U_1[x,x]\Big) +$$
$$\beta^3\,\Big(\frac{4}{3}\,\alpha^3\,\mu^2\,\hbar^4\,U_{-1}[\,]\,U_1[x] - \frac{4}{3}\,\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[x] - \frac{1}{6}\,\alpha^3\,\mu\,\hbar^4\,U_{-1}[y]\,U_1[x] -$$
$$\frac{3}{2}\,\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[\,]\,U_1[a,x] + \frac{5}{4}\,\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[a,x] + \frac{13}{6}\,\alpha^3\,\mu^3\,\hbar^5\,U_{-1}[\,]\,U_1[x,x] -$$
$$\frac{1}{2}\,\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[y]\,U_1[x,x] + \frac{1}{6}\,\alpha\,\mu\,\hbar^4\,U_{-1}[b]\,U_1[a,a,a] - \frac{1}{12}\,\alpha^2\,\mu\,\hbar^5\,U_{-1}[b,b]\,U_1[a,a,a] +$$
$$\alpha\,\mu^2\,\hbar^4\,U_{-1}[\,]\,U_1[a,a,x] - \frac{3}{4}\,\alpha^2\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[a,a,x] - \alpha^2\,\mu^3\,\hbar^5\,U_{-1}[\,]\,U_1[a,x,x]\Big) +$$
$$\beta^4\,\Big(\frac{2}{3}\,\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[\,]\,U_1[x] - \frac{1}{24}\,\alpha^4\,\mu\,\hbar^5\,U_{-1}[y]\,U_1[x] - \frac{5}{6}\,\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[\,]\,U_1[a,x] +$$
$$\frac{3}{4}\,\alpha^2\,\mu^2\,\hbar^5\,U_{-1}[\,]\,U_1[a,a,x] - \frac{1}{24}\,\alpha\,\mu\,\hbar^5\,U_{-1}[b]\,U_1[a,a,a,a] - \frac{1}{3}\,\alpha\,\mu^2\,\hbar^5\,U_{-1}[\,]\,U_1[a,a,a,x]\Big)$$

```
K = 3; $TD = 5;
GS2 = Collect[Normal@Series[GS1, {β, 0, K}], β, Factor];
(G2 = 𝕆[U₋₁[] U₁[x → x, a → a], GS2]) // DUForm
Collect[ExpandAB[
   Expand[DU₁[y] + (T - 1) μ DU₁[] + G2 - 𝕆[U₋₁[] U₁[x → x], ⅇ^(ℏ μ x)] ** DU₁[y] **
       𝕆[U₋₁[] U₁[x → x], ⅇ^(-ℏ μ x)]] /. T * ℰ_ :→ ℰ ** DU₁[A[1]] ** DU₁[B[1]]
  ],
 β]
```

$$\beta\,\mu\,\hbar\,DU_1[a] + T\,\beta\,\mu\,\hbar\,DU_1[a] + \frac{1}{2}\alpha\,\beta\,\mu^2\,\hbar^2\,DU_1[x] + \frac{1}{2}T\,\alpha\,\beta\,\mu^2\,\hbar^2\,DU_1[x] + \frac{1}{4}\alpha^2\,\beta^2\,\mu^2\,\hbar^3\,DU_1[x] - \frac{1}{4}T\,\alpha^2\,\beta^2\,\mu^2\,\hbar^3\,DU_1[x] - \frac{5}{24}\alpha^3\,\beta^3\,\mu^2\,\hbar^4\,DU_1[x] - \frac{5}{24}T\,\alpha^3\,\beta^3\,\mu^2\,\hbar^4\,DU_1[x] - \frac{1}{2}\beta^2\,\mu\,\hbar^2\,DU_1[a,a] + \frac{1}{2}T\,\beta^2\,\mu\,\hbar^2\,DU_1[a,a] - \frac{1}{2}\alpha\,\beta^2\,\mu^2\,\hbar^3\,DU_1[a,x] + \frac{1}{2}T\,\alpha\,\beta^2\,\mu^2\,\hbar^3\,DU_1[a,x] - \frac{1}{4}\alpha^2\,\beta^3\,\mu^2\,\hbar^4\,DU_1[a,x] - \frac{1}{4}T\,\alpha^2\,\beta^3\,\mu^2\,\hbar^4\,DU_1[a,x] + \frac{1}{6}\beta^3\,\mu\,\hbar^3\,DU_1[a,a,a] + \frac{1}{6}T\,\beta^3\,\mu\,\hbar^3\,DU_1[a,a,a] + \frac{1}{4}\alpha\,\beta^3\,\mu^2\,\hbar^4\,DU_1[a,a,x] + \frac{1}{4}T\,\alpha\,\beta^3\,\mu^2\,\hbar^4\,DU_1[a,a,x]$$

$$-2\,\alpha\,\mu\,\hbar\,U_{-1}[b]\,U_1[] + \alpha^2\,\mu\,\hbar^2\,U_{-1}[b,b]\,U_1[] - \frac{1}{3}\alpha^3\,\mu\,\hbar^3\,U_{-1}[b,b,b]\,U_1[] + \frac{1}{12}\alpha^4\,\mu\,\hbar^4\,U_{-1}[b,b,b,b]\,U_1[] - \frac{1}{60}\alpha^5\,\mu\,\hbar^5\,U_{-1}[b,b,b,b,b]\,U_1[] +$$
$$\beta\left(\alpha\,\mu\,\hbar^2\,U_{-1}[b]\,U_1[a] - \frac{1}{2}\alpha^2\,\mu\,\hbar^3\,U_{-1}[b,b]\,U_1[a] + \frac{1}{6}\alpha^3\,\mu\,\hbar^4\,U_{-1}[b,b,b]\,U_1[a] - \frac{1}{24}\alpha^4\,\mu\,\hbar^5\,U_{-1}[b,b,b,b]\,U_1[a] + 2\,\alpha\,\mu^2\,\hbar^2\,U_{-1}[]\,U_1[x] - 2\,\alpha^2\,\mu^2\,\hbar^3\,U_{-1}[b]\,U_1[x] - \alpha\,\mu\,\hbar^2\,U_{-1}[y]\,U_1[x] + \alpha^3\,\mu^2\,\hbar^4\,U_{-1}[b,b]\,U_1[x] - \frac{1}{3}\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b,b,b]\,U_1[x]\right) +$$
$$\beta^2\left(2\,\alpha^2\,\mu^2\,\hbar^3\,U_{-1}[]\,U_1[x] - 2\,\alpha^3\,\mu^2\,\hbar^4\,U_{-1}[b]\,U_1[x] - \frac{1}{2}\alpha^2\,\mu\,\hbar^3\,U_{-1}[y]\,U_1[x] + \alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b,b]\,U_1[x] - \frac{1}{2}\alpha\,\mu\,\hbar^3\,U_{-1}[b]\,U_1[a,a] + \frac{1}{4}\alpha^2\,\mu\,\hbar^4\,U_{-1}[b,b]\,U_1[a,a] - \frac{1}{12}\alpha^3\,\mu\,\hbar^5\,U_{-1}[b,b,b]\,U_1[a,a] - 2\,\alpha\,\mu^2\,\hbar^3\,U_{-1}[]\,U_1[a,x] + \frac{3}{2}\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[b]\,U_1[a,x] - \frac{3}{4}\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[b,b]\,U_1[a,x] + \alpha^2\,\mu^3\,\hbar^4\,U_{-1}[]\,U_1[x,x] - \frac{7}{6}\alpha^3\,\mu^3\,\hbar^5\,U_{-1}[b]\,U_1[x,x] - \frac{1}{2}\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[y]\,U_1[x,x]\right) +$$
$$\beta^3\left(\frac{3}{4}\alpha^3\,\mu^2\,\hbar^4\,U_{-1}[]\,U_1[x] - \frac{25}{24}\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[x] - \frac{1}{6}\alpha^3\,\mu\,\hbar^4\,U_{-1}[y]\,U_1[x] - \frac{3}{2}\alpha^2\,\mu^2\,\hbar^4\,U_{-1}[]\,U_1[a,x] + \frac{5}{4}\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[a,x] + \frac{11}{6}\alpha^3\,\mu^3\,\hbar^5\,U_{-1}[]\,U_1[x,x] - \frac{1}{2}\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[y]\,U_1[x,x] + \frac{1}{6}\alpha\,\mu\,\hbar^4\,U_{-1}[b]\,U_1[a,a,a] - \frac{1}{12}\alpha^2\,\mu\,\hbar^5\,U_{-1}[b,b]\,U_1[a,a,a] + \alpha\,\mu^2\,\hbar^4\,U_{-1}[]\,U_1[a,a,x] - \frac{3}{4}\alpha^2\,\mu^2\,\hbar^5\,U_{-1}[b]\,U_1[a,a,x] - \frac{7}{6}\alpha^2\,\mu^3\,\hbar^5\,U_{-1}[]\,U_1[a,x,x]\right) +$$
$$\beta^4\left(\frac{1}{12}\alpha^4\,\mu^2\,\hbar^5\,U_{-1}[]\,U_1[x] - \frac{1}{24}\alpha^4\,\mu\,\hbar^5\,U_{-1}[y]\,U_1[x] - \frac{13}{24}\alpha^3\,\mu^2\,\hbar^5\,U_{-1}[]\,U_1[a,x] + \frac{3}{4}\alpha^2\,\mu^2\,\hbar^5\,U_{-1}[]\,U_1[a,a,x] - \frac{1}{3}\alpha\,\mu^2\,\hbar^5\,U_{-1}[]\,U_1[a,a,a,x]\right) + \frac{1}{40}\beta^5\,\mu\,\hbar^5\,U_{-1}[]\,U_1[a,a,a,a,a]$$

```
Clear[ℏ]
```

```
K = 3; $TD = 5;
GS1 = μ T (A^-1 ϕ1[ħ μ (q - 1) x] - 1) - μ (A ϕ1[ħ μ (q^-1 - 1) x] - 1) /. {A → ⅇ^(-ħ β a)}
```

$$-\mu\left(-1+\frac{e^{-a\beta\hbar}\left(-1+e^{\left(-1+e^{-\alpha\beta\hbar}\right)x\mu\hbar}\right)}{\left(-1+e^{-\alpha\beta\hbar}\right)x\mu\hbar}\right)+T\mu\left(-1+\frac{e^{a\beta\hbar}\left(-1+e^{\left(-1+e^{\alpha\beta\hbar}\right)x\mu\hbar}\right)}{\left(-1+e^{\alpha\beta\hbar}\right)x\mu\hbar}\right)$$

```
K = 3;
GS2 = Collect[Normal@Series[GS1, {β, 0, K}], β, Factor]
```

$$\frac{1}{2}(1+T)\beta\mu\hbar\left(2a+x\alpha\mu\hbar\right)+\frac{1}{4}(-1+T)\beta^2\mu\hbar^2\left(2a^2+2ax\alpha\mu\hbar+x\alpha^2\mu\hbar\right)+$$
$$\frac{1}{24}(1+T)\beta^3\mu\hbar^3\left(4a^3+6a^2x\alpha\mu\hbar+6ax\alpha^2\mu\hbar-5x\alpha^3\mu\hbar\right)$$

```
Series[GS1 - GS2, {ħ, 0, $TD}]
```

$$\frac{1}{24}\left(-a^4\beta^4\mu+a^4T\beta^4\mu+7x\alpha^3\beta^3\mu^2+7Tx\alpha^3\beta^3\mu^2-4x^2\alpha^2\beta^2\mu^3+4Tx^2\alpha^2\beta^2\mu^3\right)\hbar^4+$$
$$\frac{1}{240}\left(2a^5\beta^5\mu+2a^5T\beta^5\mu-20a^3x\alpha\beta^4\mu^2+20a^3Tx\alpha\beta^4\mu^2-30a^2x\alpha^2\beta^4\mu^2+\right.$$
$$30a^2Tx\alpha^2\beta^4\mu^2-20ax\alpha^3\beta^4\mu^2+20aTx\alpha^3\beta^4\mu^2-5x\alpha^4\beta^4\mu^2+5Tx\alpha^4\beta^4\mu^2+$$
$$\left.40ax^2\alpha^2\beta^3\mu^3+40aTx^2\alpha^2\beta^3\mu^3+40x^2\alpha^3\beta^3\mu^3+40Tx^2\alpha^3\beta^3\mu^3\right)\hbar^5+O[\hbar]^6$$

```
Series[Normal@Series[GS1 - GS2, {ħ, 0, $TD}], {β, 0, K}]
```

$$\frac{1}{6}\left(-x^2\alpha^2\mu^3\hbar^4+Tx^2\alpha^2\mu^3\hbar^4\right)\beta^2+\frac{1}{24}$$
$$\left(7x\alpha^3\mu^2\hbar^4+7Tx\alpha^3\mu^2\hbar^4+4ax^2\alpha^2\mu^3\hbar^5+4aTx^2\alpha^2\mu^3\hbar^5+4x^2\alpha^3\mu^3\hbar^5+4Tx^2\alpha^3\mu^3\hbar^5\right)\beta^3+O[\beta]^4$$

```
Normal@Series[GS1 - GS2, {ħ, 0, $TD}] + O[β]^(K+1)
```

$$\frac{1}{6}\left(-x^2\alpha^2\mu^3\hbar^4+Tx^2\alpha^2\mu^3\hbar^4\right)\beta^2+\frac{1}{24}$$
$$\left(7x\alpha^3\mu^2\hbar^4+7Tx\alpha^3\mu^2\hbar^4+4ax^2\alpha^2\mu^3\hbar^5+4aTx^2\alpha^2\mu^3\hbar^5+4x^2\alpha^3\mu^3\hbar^5+4Tx^2\alpha^3\mu^3\hbar^5\right)\beta^3+O[\beta]^4$$

```
Series[GS1 - GS2, {β, 0, K}]
```

$$O[\beta]^4$$