Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-05/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as $e^{\hbar h}$).
2. dm is not meta-associative.
3. R doesn't satisfy YB.
4. Improve DUForm.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The "degree carrier" is $\hbar$, and all "coupling constants" are proportional to it.

```
$TD = 3; ℏ /: ℏ^d_. /; d > $TD := 0;
```

The 2D Lie BiAlgebra Pencil

We hope to stick to $A = e^{\hbar\beta a}$ and to $B = e^{\hbar\alpha b}$, where $[a, x] = \alpha x$ and $[b, y] = -\beta y$.
Also, $\Delta_{12}(a, A, x, b, B, y) = (a_1 + a_2,\ A_1 A_2,\ x_1 + A_1 x_2,\ b_1 + b_2,\ B_1 B_2,\ y_1 B_2 + y_2)$.
Also, (a, x) and $\hbar(b, y)$ are dual bases.

```
AlgebraAtom = a | A[_] | x | b | B[_] | y;
$PBWRule = {A[_] → 1, a → 2, x → 3, y → 4, B[_] → 5, b → 6};
```

```
B[a, x] = α x; B[x, A[n_]] = (e^(n ℏ α β) - 1) U[A[n], x]; B[a, A[_]] = 0;
B[y, b] = β y; B[B[n_], y] = (e^(n ℏ α β) - 1) U[y, B[n]]; B[b, B[_]] = 0;
Δ[a] = U1[a] U2[] + U1[] U2[a]; Δ[A[n_]] := U1[A[n]] U2[A[n]];
Δ[x] = U1[x] U2[] + U1[A[1]] U2[x];
Δ[b] = U1[b] U2[] + U1[] U2[b]; Δ[B[n_]] := U1[B[n]] U2[B[n]];
Δ[y] = U1[y] U2[B[1]] + U1[] U2[y];
S[a] = -a; S[A[n_]] := A[-n]; S[x] = - U[A[-1], x];
S[b] = -b; S[B[n_]] := B[-n]; S[y] = - U[y, B[-1]];
Si[a] = -a; Si[A[n_]] := A[-n]; Si[x] = - U[x, A[-1]];
Si[b] = -b; Si[B[n_]] := B[-n]; Si[y] = - U[B[-1], y];
(* This extra line is annoying *)
```

```
ExpandAB[ℰ_] := Expand@Normal@Series[ℰ //. {
      c_. Ui_[λ___, A[n_], ρ___] :>
        Expand[c Sum[((-1)^d ℏ^d β^d n^d)/d! Ui[λ, Sequence @@ Table[a, {d}], ρ], {d, 0, $TD}]],
      c_. Ui_[λ___, B[n_], ρ___] :> Expand[
        c Sum[((-1)^d ℏ^d α^d n^d)/d! Ui[λ, Sequence @@ Table[b, {d}], ρ], {d, 0, $TD}]]
     },
    {ℏ,
     0,
     $TD}]
```

UEA with provisional modification

This section is based on penseive://Projects/UEA/.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^p_] := U_i[x[n p]];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
     U[xs__] :→ U_i[xs],
     x : AlgebraAtom :→ U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2→i;
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k];
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j→i) U_j[yy] // Expand // m_i,j→k)
     c // USimp
  ];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :→ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
   {Sx = Supp[x], Sy = Supp[y], is, σ, z},
   If[MatchQ[Sx ⋃ Sy, {_Integer ...}] && Min[Sx ⋃ Sy] < 0,
    is = Abs[Sx] ⋂ Abs[Sy];
    z = x; Do[z = m_{-i→-σ@i}[m_{i→σ@i}[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_{σ@i,i→i}[z], {i, is}];
    z,
    (* else *) is = Sx ⋂ Sy;
    z = x; Do[z = m_{i→σ@i}[z], {i, is}];
    z = USimp[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}];
    z
   ]
  ];
UB[x_, y_] := USimp[x ** y - y ** x];
```

```
O[func_] := Normal@Series[func, {ℏ, 0, $TD}];
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] ⧴ Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ → c_) ⧴ c (
       specs /. MapThread[((#1 → _) ⧴ #3^#2) &, {vars, ps, elems}] /. U_i_ ⧴ UU_i
      )]
 ]
```

The 2D Lie BiAlgebra Pencil, Testing

```
O[U_1[a → a], e^{ℏ β a}]
```

$U_1[\,] + \beta\hbar\, U_1[a] + \frac{1}{2}\beta^2\hbar^2 U_1[a, a] + \frac{1}{6}\beta^3\hbar^3 U_1[a, a, a]$

```
USimp@With[{An = O[U_1[a → a], e^{-n ℏ β a}]}, UB[U_1[x], An] - O[e^{n ℏ α β} - 1] An ** U_1[x]]
```

0

```
B[x, A[3]]
```

$\left(-1 + e^{3\alpha\beta\hbar}\right) U[A[3], x]$

```
$TD = 6;
USimp@With[{Bn = O[U_1[b → b], e^{-n ℏ α b}]}, UB[Bn, U_1[y]] - O[e^{n ℏ α β} - 1] U_1[y] ** Bn]
```

0

```
z = U_1[a, A[2], x, x, x] U_2[a, a, x] U_3[a, a, A[-3], x];
(z // m_{1,2→1} // m_{1,3→1}) - (z // m_{2,3→2} // m_{1,2→1})
```

0

```
z = U₁[y, y, y, b, B[2]] U₂[y, b, b] U₃[y, b, b, B[-3]];
(z // m1,2→1 // m1,3→1) - (z // m2,3→2 // m1,2→1)
```

0

The Co-Product and Co-Associativity

```
Δi_→j_,k_[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
    Ui[] → Uj[] Uk[],
    Ui[f_, fs___] :→
      (USimp[(Δ[f] /. {U1 → Uj, U2 → Uk}) Δi→tj,tk[Ui[fs]]] // mj,tj→j // mk,tk→k)
  }];
Δi_→j_,k_,l__[ℰ_] := ℰ // Δi→j,k // Δk→k,l
```

```
Δ1→1,2[U1[#]] & /@ {a, A[7], x, y, b, B[-3]}
```

{U1[a] U2[] + U1[] U2[a], U1[A[7]] U2[A[7]], U1[x] U2[] + U1[A[1]] U2[x],
 U1[] U2[y] + U1[y] U2[B[1]], U1[b] U2[] + U1[] U2[b], U1[B[-3]] U2[B[-3]]}

```
{lhs = U1[x] // Δ1→1,2 // Δ2→2,3, rhs = U1[x] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
```

{U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x],
 U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x], True}

```
U1[y] // Δ1→1,2
```

U1[] U2[y] + U1[y] U2[B[1]]

```
{lhs = U1[y] // Δ1→1,2 // Δ2→2,3, rhs = U1[y] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
```

{U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]],
 U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]], True}

```
z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
```

0

```
z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
```

0

The Antipode

```
Si_[ℰ_] := Module[{ti}, USimp[
    ℰ /. Ui[x_, xs___] :→ mti,i→i[Expand[UUi[S[x]] Sti[Uti[xs]]]]
  ]];
Sii_[ℰ_] := Module[{ti}, USimp[
    ℰ /. Ui[x_, xs___] :→ mti,i→i[Expand[UUi[Si[x]] Siti[Uti[xs]]]]
  ]];
```

```
{U₁[x] // S₁ // S₁, U₁[y] // S₁ // S₁}
```

$\{e^{\alpha\beta\hbar} U_1[x], e^{\alpha\beta\hbar} U_1[y]\}$

```
{U₁[x] // S₁ // Si₁, U₁[y] // S₁ // Si₁}
```

$\{U_1[x], U_1[y]\}$

```
{z = U₁[]; (z // Δ₁→₁,₂ // S₁ // m₁,₂→₁), z = U₁[]; (z // Δ₁→₁,₂ // S₂ // m₁,₂→₁)}
```

$\{U_1[], U_1[]\}$

```
z = U₁[a, A[3], x, x];
{z // Δ₁→₁,₂ // S₂ // m₁,₂→₁, z // Δ₁→₁,₂ // S₁ // m₁,₂→₁}
```

{0, 0}

```
z = U₁[y, y, y, b, B[-3]];
{z // Δ₁→₁,₂ // S₂ // m₁,₂→₁, z // Δ₁→₁,₂ // S₁ // m₁,₂→₁}
```

{0, 0}

```
{z = U₁[a, A[2], x, x, x] U₂[a, a, A[-3], x]; (z // m₁,₂→₁ // S₁) - (z // S₁ // S₂ // m₂,₁→₁),
 z = U₁[y, y, y, b, B[2]] U₂[y, b, b, b, B[6]]; (z // m₁,₂→₁ // S₁) - (z // S₁ // S₂ // m₂,₁→₁)}
```

{0, 0}

```
{z = U₁[a, A[2], x, x, x]; (z // S₁ // Δ₁→₂,₁) - (z // Δ₁→₁,₂ // S₁ // S₂),
 z = U₁[y, y, y, b, B[2]]; (z // S₁ // Δ₁→₂,₁) - (z // Δ₁→₁,₂ // S₁ // S₂)
}
```

{0, 0}

The Pairing

```
P[U[], U[B[_]]] = P[U[A[_]], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
(    P[U[a], U[b]] = ħ^-1          P[U[a], U[B[n_]]] = -n α          P[U[a], U[y]] = 0
 P[U[A[n_]], U[b]] = -n β   P[U[A[n_]], U[B[m_]]] = e^(n m ħ α β)   P[U[A[_]], U[y]] = 0
     P[U[x], U[b]] = 0             P[U[x], U[B[_]]] = 0             P[U[x], U[y]] = ħ^-1 );
```

```
P[U[], U[]] = 1;
P_{i_,j_}[ℰ_] := USimp[ℰ /. U_i[xs___] U_j[ys___] → P[U[xs], U[ys]]];
```

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[x_, xs__], U[y_]] := P[U[x, xs], U[y]] =
   Module[{i, j, k, l}, USimp[U_i[x] UU_j[xs] Δ_{k→k,l}[U_k[y]]] // P_{i,k} // P_{j,l}];
P[U[xs__], U[y_, ys__]] := P[U[xs], U[y, ys]] =
   Module[{i, j, k, l}, USimp[Δ_{i→i,j}[UU_i[xs]] U_k[y] UU_l[ys]] // P_{i,k} // P_{j,l}];
```

```
z = Ui[a] Uj[x] Uk[y];
{mi,j→i[z] - mj,i→i[z], Δk→k,l[z] - Δk→l,k[z]}
```

```
{α Ui[x] Uk[y], -Ui[a] Uj[x] Uk[y] Ul[] +
  Ui[a] Uj[x] Uk[] Ul[y] - Ui[a] Uj[x] Uk[B[1]] Ul[y] + Ui[a] Uj[x] Uk[y] Ul[B[1]]}
```

```
Table[z = Ui[xi] Uj[xj] Uk[yk];
 {(mi,j→i[z] - mj,i→i[z]) // Pi,k, (Δk→k,l[z] - Δk→l,k[z]) // Pi,k // Pj,l},
 {xi, {a, x}}, {xj, {a, x}}, {yk, {b, y}}]
```

$\{\{\{\{0,0\},\{0,0\}\},\{\{0,0\},\{\frac{\alpha}{\hbar},\frac{\alpha}{\hbar}\}\}\},\{\{\{0,0\},\{-\frac{\alpha}{\hbar},-\frac{\alpha}{\hbar}\}\},\{\{0,0\},\{0,0\}\}\}\}$

```
Table[z = Ui[xi] Uk[yk] Ul[yl];
 {(Δi→i,j[z] - Δi→j,i[z]) // Pi,k // Pj,l, (mk,l→k[z] - ml,k→k[z]) // Pi,k},
 {xi, {a, x}}, {yk, {b, y}}, {yl, {b, y}}]
```

$\{\{\{\{0,0\},\{0,0\}\},\{\{0,0\},\{0,0\}\}\},\{\{\{0,0\},\{-\frac{\beta}{\hbar},-\frac{\beta}{\hbar}\}\},\{\{\frac{\beta}{\hbar},\frac{\beta}{\hbar}\},\{0,0\}\}\}\}$

```
lhs = Factor@Table[ℏ^n P[U @@ Table[x, {n}], U @@ Table[y, {n}]], {n, $TD = 7}]
```

$\{1,\ 1+e^{\alpha\beta\hbar},\ (1+e^{\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}),\ (1+e^{\alpha\beta\hbar})^2(1+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^2(1+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^3(1+e^{2\alpha\beta\hbar})(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})^2(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^3(1+e^{2\alpha\beta\hbar})(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})^2$
$(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}+e^{5\alpha\beta\hbar}+e^{6\alpha\beta\hbar})\}$

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^(ℏ α β)], {n, $TD = 7}]
```

$\{1,\ 1+e^{\alpha\beta\hbar},\ (1+e^{\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}),\ (1+e^{\alpha\beta\hbar})^2(1+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^2(1+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^3(1+e^{2\alpha\beta\hbar})(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})^2(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}),$
$(1+e^{\alpha\beta\hbar})^3(1+e^{2\alpha\beta\hbar})(1-e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar})^2$
$(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar})(1+e^{\alpha\beta\hbar}+e^{2\alpha\beta\hbar}+e^{3\alpha\beta\hbar}+e^{4\alpha\beta\hbar}+e^{5\alpha\beta\hbar}+e^{6\alpha\beta\hbar})\}$

```
MapThread[Equal, {lhs, rhs}]
```

```
{True, True, True, True, True, True, True}
```

```
P[U[a, a, a, a, a], U[b, b, b, b, b]]
```

$\frac{120}{\hbar^5}$

```
Table[P[z1, z2],
  {z1, {U[], U[a], U[x], U[a, a], U[a, x], U[x, x],
    U[a, a, a], U[a, a, x], U[a, x, x], U[x, x, x]}}, {z2, {U[], U[b], U[y],
    U[b, b], U[y, b], U[y, y], U[b, b, b], U[y, b, b], U[y, y, b], U[y, y, y]}}
 ] //
 MatrixForm
```

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\hbar} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\hbar} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{\hbar^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\hbar^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^2}+\frac{e^{\alpha\beta\hbar}}{\hbar^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{\hbar^3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\hbar^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^3}+\frac{e^{\alpha\beta\hbar}}{\hbar^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\hbar^3}+\frac{2e^{\alpha\beta\hbar}}{\hbar^3}+\frac{2e^{2\alpha\beta\hbar}}{\hbar^3}+\frac{e^{3\alpha\beta\hbar}}{\hbar^3}
\end{pmatrix}$$

The Double

$\text{DUForm}[\mathcal{E}\_] := \mathcal{E}\ \text{//.}\ U_{i\_}[fs\_\_\_]\ U_{j\_}[gs\_\_\_]\ \text{/;}\ i+j == 0 :\rightarrow \text{DU}_{\text{Abs}@i}[fs, gs];$

$\text{dm}_{i\_,j\_\to k\_}[\mathcal{E}\_] := \text{Module}[\{t1, t2, t3, h1, h2, h3\},$
$\quad \mathcal{E}\ \text{//}\ \Delta_{i\to h1,h2,h3}\ \text{//}\ S_{h3}\ \text{//}\ \Delta_{-j\to t1,t2,t3}\ \text{//}\ P_{h1,t1}\ \text{//}\ P_{h3,t3}\ \text{//}\ m_{j,h2\to k}\ \text{//}\ m_{-i,t2\to -k}];$
$\text{d}\Delta_{i\_\to j\_,k\_}[\mathcal{E}\_] := \mathcal{E}\ \text{//}\ \Delta_{i\to j,k}\ \text{//}\ \Delta_{-i\to -j,-k};$

$U_{-1}[]\ U_1[a]\ U_{-2}[b]\ U_2[]\ \text{//}\ \text{dm}_{1,2\to 1}$

$U_{-1}[b]\ U_1[a]$

$\{U_{-1}[]\ U_1[a]\ U_{-2}[y]\ U_2[]\ \text{//}\ \text{dm}_{2,1\to 1},\ U_{-1}[]\ U_1[a]\ U_{-2}[y]\ U_2[]\ \text{//}\ \text{dm}_{1,2\to 1}\}$

$\{U_{-1}[y]\ U_1[a],\ \alpha\, U_{-1}[y]\ U_1[] + U_{-1}[y]\ U_1[a]\}$

$U_{-1}[]\ U_1[A[1]]\ U_{-2}[y]\ U_2[]\ \text{//}\ \text{dm}_{1,2\to 1}$

$e^{-\alpha\beta\hbar}\ U_{-1}[y]\ U_1[A[1]]$

$U_{-1}[]\ U_1[x]\ U_{-2}[b]\ U_2[]\ \text{//}\ \text{dm}_{1,2\to 1}$

$-\beta\, U_{-1}[]\ U_1[x] + U_{-1}[b]\ U_1[x]$

$U_{-1}[]\ U_1[x]\ U_{-2}[B[1]]\ U_2[]\ \text{//}\ \text{dm}_{1,2\to 1}$

$e^{\alpha\beta\hbar}\ U_{-1}[B[1]]\ U_1[x]$

```
U₋₁[] U₁[x] U₋₂[y] U₂[] // dm_{1,2→1}
```

$$\frac{U_{-1}[B[1]]\,U_1[]}{\hbar} + U_{-1}[y]\,U_1[x] - \frac{U_{-1}[]\,U_1[A[1]]}{\hbar}$$

```
z = U₋₁[] U₁[x] U₋₂[y] U₂[] U₋₃[b] U₃[];
(z // dm_{1,2→1} // dm_{1,3→1}) - (z // dm_{2,3→2} // dm_{1,2→1}) // DUForm
```

0

```
z = U₋₁[b, b, y] U₁[A[2], x, x] U₋₂[B[-1], y, y] U₂[a] U₋₃[b, y] U₃[a, a, x];
(z // dm_{1,2→1} // dm_{1,3→1}) - (z // dm_{2,3→2} // dm_{1,2→1}) // DUForm
```

0

The R-Matrix

Using Quesne's formula.

```
R_{i_,j_} := 𝕆[
  U₋ᵢ[y → y, b → b] Uᵢ[] U₋ⱼ[] Uⱼ[a → a, x → x],
  Series[Exp[ℏ b a + Sum_{k=1}^{$TD} ((1 - e^{ℏ α β})^k (ℏ x y)^k) / (k (1 - e^{k ℏ α β}))], {ℏ, 0, $TD}]
 ]
```

```
$TD = 3; R_{1,2} // DUForm
```

$$DU_1[]\,DU_2[] + \hbar\,DU_1[b]\,DU_2[a] + \hbar\,DU_1[y]\,DU_2[x] + \frac{1}{2}\hbar^2\,DU_1[b,b]\,DU_2[a,a] + \hbar^2\,DU_1[y,b]\,DU_2[a,x] + \frac{1}{2}\hbar^2\,DU_1[y,y]\,DU_2[x,x] - \frac{1}{4}\alpha\beta\hbar^3\,DU_1[y,y]\,DU_2[x,x] + \frac{1}{6}\hbar^3\,DU_1[b,b,b]\,DU_2[a,a,a] + \frac{1}{2}\hbar^3\,DU_1[y,b,b]\,DU_2[a,a,x] + \frac{1}{2}\hbar^3\,DU_1[y,y,b]\,DU_2[a,x,x] + \frac{1}{6}\hbar^3\,DU_1[y,y,y]\,DU_2[x,x,x]$$

```
$TD = 5; {lhs = R_{1,3} // dΔ_{1→1,2}, rhs = R_{2,3} ** R_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm
```

0

```
$TD = 5; {lhs = R_{1,2} // dΔ_{2→2,3}, rhs = R_{1,2} ** R_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm
```

0

```
$TD = 5; Table[
 {lhs = R_{1,2} ** dΔ_{1→2,1}[f], rhs = dΔ_{1→1,2}[f] ** R_{1,2}, lhs - rhs} // Last // ExpandAB // DUForm,
 {f, {U₋₁[] U₁[a], U₋₁[] U₁[x], U₋₁[b] U₁[], U₋₁[y] U₁[]}}]
```

{0, 0, 0, 0}

```
$TD = 5;
{lhs = ExpandAB[R_{1,2} ** R_{1,3} ** R_{2,3}],
  rhs = ExpandAB[R_{2,3} ** R_{1,3} ** R_{1,2}], Coefficient[lhs - rhs, ℏ^{$TD}]} // Last
```

0

```
$TD = 5; Si₂[R₁,₂] ** R₁,₂ // ExpandAB // DUForm
```

$DU_1[\,]\,DU_2[\,]$

Cuaps

Drinfeld element u

```
u_i_ := R₁,₂ // S₂ // m₂,₁→i
```

```
$TD = 1; u₁ // DUForm
```

$DU_1[\,]\,U_{-2}[\,] - \hbar\,DU_1[b, a]\,U_{-2}[\,] - \hbar\,DU_1[y, A[-1], x]\,U_{-2}[\,]$

```
$TD = 1; Table[
 {lhs = f ** u₁, rhs = u₁ ** S₁[S₁[f]], lhs - rhs} // Last // ExpandAB // DUForm,
 {f, {U₋₁[] U₁[a], U₋₁[] U₁[x], U₋₁[b] U₁[], U₋₁[y] U₁[]}}]
```

$\{0, 0, 0, -\alpha\,\beta\,\hbar\,DU_1[y]\,U_{-2}[\,]\}$