

Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-05/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as $e^{\hbar h}$).
2. dm is not meta-associative.
3. R doesn't satisfy YB.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The “degree carrier” is \hbar , and all “coupling constants” are proportional to it.

```
$TD = 3;  $\hbar$  /:  $\hbar^{d-}$  /;  $d > $TD := 0;$ 
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\hbar \epsilon g}$ and to $H = e^{\hbar \gamma h}$, where $[g, e] = \gamma e$ and $[h, f] = -\epsilon f$.

Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2, G_1 G_2, e_1 + G_1 e_2, h_1 + h_2, H_1 H_2, f_1 H_2 + f_2)$.

Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;
$PBWRule = {G[_] → 1, g → 2, e → 3, H[_] → 4, h → 5, f → 6};
```

```
B[g, e] =  $\gamma e$ ; B[e, G[n_]] = (e-n $\hbar$  $\gamma e$  - 1) U[G[n], e]; B[g, G[_]] = 0;
B[h, f] = - $\epsilon f$ ; B[f, H[n_]] = (en $\hbar$  $\gamma e$  - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on pensieve://Projects/UEA/.

```
B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[# , y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```

x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^p_] := U_i[x[n p]];
UU_i_[x_^p_] := UU_i@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];

```

```

m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_i->j_[ε_] := ε /. U_i -> U_j;

```

```

m_i->j->k_[c_. U_i[x_] U_j[]] := c U_k[x];
m_i->j->k_[c_. U_i[] U_j[y_]] := c U_k[y];
m_i->j->k_[c_. U_i[xx_, x_[n1_]] U_j[x_[n2_], yy_]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j->k];
m_i->j->k_[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];

```

```

Supp[ε_] := Union@Cases[{ε}, U_i[___] -> i, ∞];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
  {Sx = Supp[x], Sy = Supp[y], is, σ, z},
  If[MatchQ[Sx ∪ Sy, {_Integer ...}] && Min[Sx ∪ Sy] < 0,
    is = Abs[Sx] ∩ Abs[Sy];
    z = x; Do[z = m_i->σ@i[m_i->σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_σ@i,i->i[z], {i, is}];
    z,
    (* else *) is = Sx ∩ Sy;
    z = x; Do[z = m_i->σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = m_σ@i,i->i[z], {i, is}];
    z
  ]
];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[func_] := Normal@Series[func, {h, 0, $TD}];
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U[u___] => Cases[{u}, r_Rule], infinity];
  vars = First/@rules; elems = Last/@rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ -> c_) => c (
    specs /. MapThread[{(u1 -> _) => u2} &, {vars, ps, elems}] /. U_i_ => UU_i
  )]
]

```

The 2D Lie BiAlgebra Pencil, Testing

$$O[U_1[x \to g], e^{h \epsilon x}]$$

$$U_1[] + \epsilon \hbar U_1[g] + \frac{1}{2} \epsilon^2 \hbar^2 U_1[g, g] + \frac{1}{6} \epsilon^3 \hbar^3 U_1[g, g, g]$$

$$USimp@With[{G = O[U_1[x \to g], e^{h \epsilon x}]}, UB[U_1[e], G] - O[e^{-h \gamma \epsilon} - 1] G ** U_1[e]}$$

0

$$B[e, G[3]]$$

$$(-1 + e^{-3 \gamma \epsilon \hbar}) U[G[3], e]$$

$$\$TD = 6;$$

$$USimp@With[{H = O[U_1[x \to h], e^{h \gamma x}]}, UB[U_1[f], H] - O[e^{h \gamma \epsilon} - 1] H ** U_1[f]}$$

0

$$x = U_1[g, G[2], e, e, e] U_2[g, g, e] U_3[g, g, G[-3], e];$$

$$(x // m_{1,2 \to 1} // m_{1,3 \to 1}) - (x // m_{2,3 \to 2} // m_{1,2 \to 1})$$

0

$$x = U_1[h, H[2], f, f, f] U_2[h, h, f] U_3[h, h, H[-3], f];$$

$$(x // m_{1,2 \to 1} // m_{1,3 \to 1}) - (x // m_{2,3 \to 2} // m_{1,2 \to 1})$$

0

The Co-Product and Co-Associativity

```

qΔi→j,k[ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[g, xs---] :=>
    (USimp[(Uj[g] Uk[] + Uj[] Uk[g]) qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k),
  Ui[G[n-], xs---] :=> (USimp[Uj[G[n]] Uk[G[n]] qΔi→tj,tk[Ui[xs]]] //
    mj,tj→j // mk,tk→k),
  Ui[e, xs---] :=> (USimp[(Uj[e] Uk[G[1]] + Uj[] Uk[e]) qΔi→tj,tk[Ui[xs]]] //
    mj,tj→j // mk,tk→k),
  Ui[h, xs---] :=> (USimp[(Uj[h] Uk[] + Uj[] Uk[h]) qΔi→tj,tk[Ui[xs]]] //
    mj,tj→j // mk,tk→k),
  Ui[H[n-], xs---] :=> (USimp[Uj[H[n]] Uk[H[n]] qΔi→tj,tk[Ui[xs]]] //
    mj,tj→j // mk,tk→k),
  Ui[f, xs---] :=> (USimp[(Uj[f] Uk[] + Uj[H[1]] Uk[f]) qΔi→tj,tk[Ui[xs]]] //
    mj,tj→j // mk,tk→k)
}];

```

```

qΔi→j,k,l[ε-] := ε // qΔi→j,k // qΔk→l

```

```

U1[e] // qΔ1→1,2
U1[] U2[e] + U1[e] U2[G[1]]

{lhs = U1[e] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[e] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
{U1[] U2[] U3[e] + U1[] U2[e] U3[G[1]] + U1[e] U2[G[1]] U3[G[1]],
 U1[] U2[] U3[e] + U1[] U2[e] U3[G[1]] + U1[e] U2[G[1]] U3[G[1]], True}

U1[f] // qΔ1→1,2
U1[f] U2[] + U1[H[1]] U2[f]

{lhs = U1[f] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[f] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
{U1[f] U2[] U3[] + U1[H[1]] U2[f] U3[] + U1[H[1]] U2[H[1]] U3[f],
 U1[f] U2[] U3[] + U1[H[1]] U2[f] U3[] + U1[H[1]] U2[H[1]] U3[f], True}

x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // qΔ1→1,2) - (x // qΔ2→3,4 // qΔ1→1,2 // m1,3→1 // m2,4→2)
0

x = U1[h, H[2], f, f, f] U2[h, h, H[-3], f];
(x // m1,2→1 // qΔ1→1,2) - (x // qΔ2→3,4 // qΔ1→1,2 // m1,3→1 // m2,4→2)
0

```

The Antipode

Why o why this annoyance of left-vs-right?

```

S[g] = -g; S[G[n_]] := G[-n]; S[e] = -eh̄γ U[G[-1], e];
S[h] = -h; S[H[n_]] := H[-n]; S[f] = -U[H[-1], f];
S_i[ε_] := Module[{ti}, USimp[
  ε /. U_i[x_, xs_] => mti,i[Expand[UU_i[S[x]] Sti[Uti[xs]]]]
]];

```

```
{lhs = S1[U1[e]], rhs = -U1[e] ** U1[G[-1]], lhs == rhs}
```

```
{-eh̄γ U1[G[-1], e], -eh̄γ U1[G[-1], e], True}
```

```
U1[e] // S1 // S1
```

```
eh̄γ U1[e]
```

```
U1[f] // S1 // S1
```

```
eh̄γ U1[f]
```

```
S1[U1[g, G[3], e, e]]
```

```
2 e9h̄γ U1[G[-5], e, e] - e9h̄γ U1[G[-5], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2
```

```
U1[G[3], g, e, e] U2[G[5]] + U1[G[3], g, e] U2[G[4], e] + e-h̄γ U1[G[3], g, e] U2[G[4], e] +
U1[G[3], e, e] U2[G[5], g] + U1[G[3], g] U2[G[3], e, e] + U1[G[3], e] U2[G[4], g, e] +
e-h̄γ U1[G[3], e] U2[G[4], g, e] + U1[G[3]] U2[G[3], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2 // S2
```

```
U1[G[3], g, e, e] U2[G[-5]] -
e4h̄γ U1[G[3], e] U2[G[-5], e] - e5h̄γ U1[G[3], e] U2[G[-5], e] -
e4h̄γ U1[G[3], g, e] U2[G[-5], e] - e5h̄γ U1[G[3], g, e] U2[G[-5], e] -
U1[G[3], e, e] U2[G[-5], g] + 2 e9h̄γ U1[G[3]] U2[G[-5], e, e] +
e9h̄γ U1[G[3], g] U2[G[-5], e, e] + e4h̄γ U1[G[3], e] U2[G[-5], g, e] +
e5h̄γ U1[G[3], e] U2[G[-5], g, e] - e9h̄γ U1[G[3]] U2[G[-5], g, e, e]
```

```
test = U1[g, G[3], e, e];
```

```
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
```

```
test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, 2 e6h̄γ U1[e, e] - 2 e7h̄γ U1[e, e] - 2 e8h̄γ U1[e, e] + 2 e9h̄γ U1[e, e]}
```

```
test = U1[h, H[3], f, f];
```

```
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
```

```
test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, -2 e-10h̄γ U1[H[-2], f, f] + 2 e-9h̄γ U1[H[-2], f, f] +
2 e-8h̄γ U1[H[-2], f, f] - 2 e-7h̄γ U1[H[-2], f, f]}
```

```
x = U1[h, H[2], f, f] U2[h, h, H[-3], f];
```

```
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
```

```
0
```

```
x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
0
```

```
x = U1[];
(x // qΔ1→1,2 // S1 // m1,2→1)
U1[]
```

```
x = U1[];
(x // qΔ1→1,2 // S2 // m1,2→1)
U1[]
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[]] = 1;
P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1;
P[U[], U[[_]]] = P[U[[_]], U[]] = 0;
(
  P[U[g], U[h]] = ħ-1      P[U[g], U[H[n_]]] = n γ      P[U[g], U[f]] = 0
  P[U[G[n_]], U[h]] = n ε  P[U[G[n_]], U[H[m_]]] = en m ħ γ ε P[U[G[_]], U[f]] = 0
  P[U[e], U[h]] = 0        P[U[e], U[H[_]]] = 0          P[U[e], U[f]] = ħ-1
);
```

```
Pi,j[ε] := USimp[ε /. Ui[xs___] Uj[ys___] → P[U[xs], U[ys]]];
```

```
t = Ui[g] Uj[e] Uk[f];
{mi,j→i[t] - mj,i→i[t], qΔk→k,1[t] - qΔk→1,k[t]}
{∑ Ui[e] Uk[f], Ui[g] Uj[e] Uk[f] U1[] - Ui[g] Uj[e] Uk[] U1[f] +
  Ui[g] Uj[e] Uk[H[1]] U1[f] - Ui[g] Uj[e] Uk[f] U1[H[1]]}
```

$$\begin{aligned}
 & \mathbf{t} = \mathbf{U}_i[\mathbf{g}] \mathbf{U}_j[\mathbf{e}] \mathbf{U}_k[\mathbf{f}]; \\
 & \{(\mathbf{m}_{i,j \rightarrow i}[\mathbf{t}] - \mathbf{m}_{j,i \rightarrow i}[\mathbf{t}]) // \mathbf{P}_{i,k}, (\mathbf{q}_{\Delta_{k \rightarrow k,1}}[\mathbf{t}] - \mathbf{q}_{\Delta_{k \rightarrow 1,k}}[\mathbf{t}]) // \mathbf{P}_{i,k} // \mathbf{P}_{j,1}\} \\
 & \left\{ \frac{\gamma}{\hbar}, \frac{\gamma}{\hbar} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Table}[\mathbf{t} = \mathbf{U}_i[\mathbf{x}_i] \mathbf{U}_j[\mathbf{x}_j] \mathbf{U}_k[\mathbf{y}_k]; \\
 & \quad \{(\mathbf{m}_{i,j \rightarrow i}[\mathbf{t}] - \mathbf{m}_{j,i \rightarrow i}[\mathbf{t}]) // \mathbf{P}_{i,k}, (\mathbf{q}_{\Delta_{k \rightarrow k,1}}[\mathbf{t}] - \mathbf{q}_{\Delta_{k \rightarrow 1,k}}[\mathbf{t}]) // \mathbf{P}_{i,k} // \mathbf{P}_{j,1}\}, \\
 & \quad \{\mathbf{x}_i, \{\mathbf{g}, \mathbf{e}\}\}, \{\mathbf{x}_j, \{\mathbf{g}, \mathbf{e}\}\}, \{\mathbf{y}_k, \{\mathbf{h}, \mathbf{f}\}\}] \\
 & \{ \{ \{ \{ \{ \mathbf{0}, \mathbf{0} \}, \{ \mathbf{0}, \mathbf{0} \} \}, \{ \{ \mathbf{0}, \mathbf{0} \}, \{ \frac{\gamma}{\hbar}, \frac{\gamma}{\hbar} \} \} \}, \{ \{ \{ \mathbf{0}, \mathbf{0} \}, \{ -\frac{\gamma}{\hbar}, -\frac{\gamma}{\hbar} \} \}, \{ \{ \mathbf{0}, \mathbf{0} \}, \{ \mathbf{0}, \mathbf{0} \} \} \} \} \}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Table}[\mathbf{t} = \mathbf{U}_i[\mathbf{x}_i] \mathbf{U}_k[\mathbf{y}_k] \mathbf{U}_j[\mathbf{y}_1]; \\
 & \quad \{(\mathbf{q}_{\Delta_{i \rightarrow i,j}}[\mathbf{t}] - \mathbf{q}_{\Delta_{i \rightarrow j,i}}[\mathbf{t}]) // \mathbf{P}_{i,k} // \mathbf{P}_{j,1}, (\mathbf{m}_{k,1 \rightarrow k}[\mathbf{t}] - \mathbf{m}_{1,k \rightarrow k}[\mathbf{t}]) // \mathbf{P}_{i,k}\}, \\
 & \quad \{\mathbf{x}_i, \{\mathbf{g}, \mathbf{e}\}\}, \{\mathbf{y}_k, \{\mathbf{h}, \mathbf{f}\}\}, \{\mathbf{y}_1, \{\mathbf{h}, \mathbf{f}\}\}] \\
 & \{ \{ \{ \{ \{ \mathbf{0}, \mathbf{0} \}, \{ \mathbf{0}, \mathbf{0} \} \}, \{ \{ \mathbf{0}, \mathbf{0} \}, \{ \mathbf{0}, \mathbf{0} \} \} \}, \{ \{ \{ \mathbf{0}, \mathbf{0} \}, \{ -\frac{\epsilon}{\hbar}, -\frac{\epsilon}{\hbar} \} \}, \{ \{ \frac{\epsilon}{\hbar}, \frac{\epsilon}{\hbar} \}, \{ \mathbf{0}, \mathbf{0} \} \} \} \} \}
 \end{aligned}$$

General Pairings

The pairing sequence: (one,one) (above), (many,one), (many,many).

```

P[U[x_, xs_], U[y_]] := P[U[x, xs], U[y]] =
  Module[{i, j, k, l}, USimp[U_i[x] U_j[xs] q_{\Delta_{k \to k,1}}[U_k[y]]] // P_{i,k} // P_{j,1};
P[U[xs_], U[y_, ys_]] := P[U[xs], U[y, ys]] =
  Module[{i, j, k, l}, USimp[q_{\Delta_{i \to i,j}}[U_i[xs]] U_k[y] U_l[ys]] // P_{i,k} // P_{j,1};

```

$$\begin{aligned}
 & \{P[U[\mathbf{g}, \mathbf{e}], U[\mathbf{h}]], P[U[\mathbf{g}, \mathbf{e}], U[\mathbf{f}]], P[U[\mathbf{e}, \mathbf{e}], U[\mathbf{f}]]\} \\
 & \left\{ \mathbf{0}, \frac{\gamma}{\hbar}, \mathbf{0} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & P[U[\mathbf{e}], U[\mathbf{f}, \mathbf{f}]] \\
 & \mathbf{0}
 \end{aligned}$$

$$\begin{aligned}
 & P[U[\mathbf{e}, \mathbf{e}], U[\mathbf{f}, \mathbf{f}]] \\
 & \frac{1}{\hbar^2} + \frac{e^{\gamma \epsilon \hbar}}{\hbar^2}
 \end{aligned}$$

$$\begin{aligned}
 & \text{lhs} = \text{Factor@Table}[\hbar^n P[U@@\text{Table}[\mathbf{e}, \{n\}], U@@\text{Table}[\mathbf{f}, \{n\}]], \{n, \$TD = 7\}] \\
 & \{1, 1 + e^{\gamma \epsilon \hbar}, (1 + e^{\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar}), (1 + e^{\gamma \epsilon \hbar})^2 (1 + e^{2\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar}), \\
 & (1 + e^{\gamma \epsilon \hbar})^2 (1 + e^{2\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar} + e^{3\gamma \epsilon \hbar} + e^{4\gamma \epsilon \hbar}), \\
 & (1 + e^{\gamma \epsilon \hbar})^3 (1 + e^{2\gamma \epsilon \hbar}) (1 - e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar})^2 (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar} + e^{3\gamma \epsilon \hbar} + e^{4\gamma \epsilon \hbar}), \\
 & (1 + e^{\gamma \epsilon \hbar})^3 (1 + e^{2\gamma \epsilon \hbar}) (1 - e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar})^2 \\
 & (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar} + e^{3\gamma \epsilon \hbar} + e^{4\gamma \epsilon \hbar}) (1 + e^{\gamma \epsilon \hbar} + e^{2\gamma \epsilon \hbar} + e^{3\gamma \epsilon \hbar} + e^{4\gamma \epsilon \hbar} + e^{5\gamma \epsilon \hbar} + e^{6\gamma \epsilon \hbar}) \}
 \end{aligned}$$

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, eħγϵ], {n, $TD = 7}]
{1, 1 + eγϵħ, (1 + eγϵħ) (1 + eγϵħ + e2γϵħ), (1 + eγϵħ)2 (1 + e2γϵħ) (1 + eγϵħ + e2γϵħ),
(1 + eγϵħ)2 (1 + e2γϵħ) (1 + eγϵħ + e2γϵħ) (1 + eγϵħ + e2γϵħ + e3γϵħ + e4γϵħ),
(1 + eγϵħ)3 (1 + e2γϵħ) (1 - eγϵħ + e2γϵħ) (1 + eγϵħ + e2γϵħ)2 (1 + eγϵħ + e2γϵħ + e3γϵħ + e4γϵħ),
(1 + eγϵħ)3 (1 + e2γϵħ) (1 - eγϵħ + e2γϵħ) (1 + eγϵħ + e2γϵħ)2
(1 + eγϵħ + e2γϵħ + e3γϵħ + e4γϵħ) (1 + eγϵħ + e2γϵħ + e3γϵħ + e4γϵħ + e5γϵħ + e6γϵħ) }
```

```
MapThread[Equal, {lhs, rhs}]
{True, True, True, True, True, True, True}
```

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

$$\frac{120}{\hbar^5}$$

```
P[U[g, g, g, g, g, e, e, e, e], U[h, h, h, h, h, f, f, f, f]] // Factor
```

$$\frac{120 (1 + e^{\gamma\epsilon\hbar})^2 (1 + e^{2\gamma\epsilon\hbar}) (1 + e^{\gamma\epsilon\hbar} + e^{2\gamma\epsilon\hbar})}{\hbar^9}$$

```
x = U1[g, G[-3], e, e] U2[g, g, G[1], e] U3[h, H[2], f, f, f];
(x // m1,2→1 // P1,3) - (x // qΔ3→3,4 // P1,3 // P2,4)
0
```

```
x = U1[h, H[-3], f, f] U2[h, h, H[1], f] U3[g, G[2], e, e, e];
(x // m1,2→1 // P3,1) - (x // qΔ3→3,4 // P3,1 // P4,2)
0
```

```
x = U1[g, g, G[-3], e, e, e] U2[h, h, H[2], f, f, f];
(x // S1 // P1,2) - (x // S2 // P1,2)
0
```

The Double

```
dmi,j→k[ε-] := Module[{t1, t2, t3, h1, h2, h3},
ε // qΔi→h1,h2,h3 // Sh1 // qΔ-j→t1,t2,t3 // Ph1,t1 // Ph3,t3 // mh2,j→k // m-i,t2→-k]
```

```
U-1[[]] U1[g] U-2[h] U2[[]] // dm1,2→1
```

$$U_{-1}[h] U_1[g]$$

```
U-1[[]] U1[g] U-2[f] U2[[]] // dm1,2→1
```

$$-\gamma U_{-1}[f] U_1[] + U_{-1}[f] U_1[g]$$

```
U-1[[]] U1[G[1]] U-2[f] U2[[]] // dm1,2→1
```

$$e^{-\gamma\epsilon\hbar} U_{-1}[f] U_1[G[1]]$$

$$U_{-1}[] U_1[e] U_{-2}[h] U_2[] // dm_{1,2 \rightarrow 1}$$

$$\in U_{-1}[] U_1[e] + U_{-1}[h] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[H[1]] U_2[] // dm_{1,2 \rightarrow 1}$$

$$e^{\gamma \in \hbar} U_{-1}[H[1]] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[f] U_2[] // dm_{1,2 \rightarrow 1}$$

$$\frac{U_{-1}[H[1]] U_1[]}{\hbar} + U_{-1}[f] U_1[e] - \frac{U_{-1}[] U_1[G[1]]}{\hbar}$$

$$x = U_{-1}[] U_1[e] U_{-2}[f] U_2[] U_{-3}[h] U_3[];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$- \frac{\in U_{-1}[H[1]] U_1[]}{\hbar} + \frac{\in U_{-1}[] U_1[G[1]]}{\hbar}$$

$$x = U_{-1}[h, h, f] U_1[G[2], e, e] U_{-2}[H[-1], f, f] U_2[g] U_{-3}[h, f] U_3[g, g, e];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$\frac{2 e^{-5 \gamma \in \hbar} \gamma^3 \in U_{-1}[h, h, f, f, f] U_1[G[2], e, e]}{\hbar} -$$

$$\frac{4 e^{-4 \gamma \in \hbar} \gamma^3 \in U_{-1}[h, h, f, f, f] U_1[G[2], e, e]}{\hbar} + \frac{2 e^{-3 \gamma \in \hbar} \gamma^3 \in U_{-1}[h, h, f, f, f] U_1[G[2], e, e]}{\hbar} +$$

$$\frac{8 e^{-2 \gamma \in \hbar} \gamma^3 \in U_{-1}[h, h, f, f, f] U_1[G[2], e, e]}{\hbar} - \frac{2 e^{-4 \gamma \in \hbar} \gamma^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e]}{\hbar} +$$

$$\frac{2 e^{-2 \gamma \in \hbar} \gamma^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e]}{\hbar} - \frac{1}{\hbar}$$

$$2 e^{-9 \gamma \in \hbar} \gamma^3 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] + \frac{1}{\hbar}$$

$$4 e^{-8 \gamma \in \hbar} \gamma^3 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] - \frac{1}{\hbar}$$

$$2 e^{-7 \gamma \in \hbar} \gamma^3 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] - \frac{1}{\hbar}$$

$$8 e^{-6 \gamma \in \hbar} \gamma^3 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] + \frac{1}{\hbar}$$

$$2 e^{-8 \gamma \in \hbar} \gamma^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] - \frac{1}{\hbar}$$

$$2 e^{-6 \gamma \in \hbar} \gamma^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] - \frac{1}{\hbar}$$

$$5 e^{-5 \gamma \in \hbar} \gamma^2 \in U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] + \frac{1}{\hbar}$$

$$10 e^{-4 \gamma \in \hbar} \gamma^2 \in U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] - \frac{1}{\hbar}$$

$$5 e^{-3 \gamma \in \hbar} \gamma^2 \in U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] - \frac{1}{\hbar}$$

$$20 e^{-2 \gamma \in \hbar} \gamma^2 \in U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] + \frac{1}{\hbar}$$

$$\begin{aligned}
 & 5 e^{-4\gamma\epsilon\hbar} \gamma^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] - \frac{1}{\hbar} \\
 & 5 e^{-2\gamma\epsilon\hbar} \gamma^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] + \frac{\gamma \in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e]}{\hbar^2} + \\
 & \frac{1}{\hbar^2} 4 e^{\gamma\epsilon\hbar} \gamma \in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] + \frac{1}{\hbar^2} \\
 & 3 e^{2\gamma\epsilon\hbar} \gamma \in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] - \\
 & \frac{\gamma U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e]}{\hbar^2} - \frac{1}{\hbar^2} \\
 & e^{-\gamma\epsilon\hbar} \gamma U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \frac{1}{\hbar^2} \\
 & e^{\gamma\epsilon\hbar} \gamma U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \frac{1}{\hbar^2} \\
 & e^{2\gamma\epsilon\hbar} \gamma U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \frac{1}{\hbar} \\
 & 5 e^{-9\gamma\epsilon\hbar} \gamma^2 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] - \frac{1}{\hbar} \\
 & 10 e^{-8\gamma\epsilon\hbar} \gamma^2 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] + \frac{1}{\hbar} \\
 & 5 e^{-7\gamma\epsilon\hbar} \gamma^2 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] + \frac{1}{\hbar} \\
 & 20 e^{-6\gamma\epsilon\hbar} \gamma^2 \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] - \frac{1}{\hbar} \\
 & 5 e^{-8\gamma\epsilon\hbar} \gamma^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] + \frac{1}{\hbar} \\
 & 5 e^{-6\gamma\epsilon\hbar} \gamma^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] - \\
 & \frac{3 \gamma \in U_{-1}[h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} - \frac{e^{-3\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} - \\
 & \frac{5 e^{-2\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} - \frac{7 e^{-\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} - \\
 & \frac{\gamma U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} + \frac{e^{-4\gamma\epsilon\hbar} \gamma U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} + \\
 & \frac{2 e^{-3\gamma\epsilon\hbar} \gamma U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} - \frac{2 e^{-\gamma\epsilon\hbar} \gamma U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e]}{\hbar^2} + \\
 & \frac{1}{\hbar^2} e^{-5\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] + \frac{1}{\hbar^2} \\
 & 4 e^{-4\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] + \frac{1}{\hbar^2} \\
 & 3 e^{-3\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] - \frac{1}{\hbar^2} \\
 & e^{-6\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] - \frac{1}{\hbar^2} \\
 & e^{-5\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \frac{1}{\hbar^2} \\
 & e^{-4\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \frac{1}{\hbar^2}
 \end{aligned}$$

$$\begin{aligned}
 & e^{-3\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \frac{1}{\hbar} \\
 & 4 e^{-5\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \frac{1}{\hbar} \\
 & 8 e^{-4\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \frac{1}{\hbar} \\
 & 4 e^{-3\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \frac{1}{\hbar} \\
 & 16 e^{-2\gamma\epsilon\hbar} \gamma \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \frac{1}{\hbar} \\
 & 4 e^{-4\gamma\epsilon\hbar} \gamma U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] + \frac{1}{\hbar} \\
 & 4 e^{-2\gamma\epsilon\hbar} \gamma U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & \frac{\in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e]}{\hbar^2} - \frac{1}{\hbar^2} \\
 & 4 e^{\gamma\epsilon\hbar} \in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2} \\
 & 3 e^{2\gamma\epsilon\hbar} \in U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] + \\
 & \frac{U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e]}{\hbar^2} + \frac{1}{\hbar^2} \\
 & e^{-\gamma\epsilon\hbar} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2} \\
 & e^{\gamma\epsilon\hbar} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2} \\
 & e^{2\gamma\epsilon\hbar} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \frac{1}{\hbar} \\
 & 4 e^{-9\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \frac{1}{\hbar} \\
 & 8 e^{-8\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \frac{1}{\hbar} \\
 & 4 e^{-7\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \frac{1}{\hbar} \\
 & 16 e^{-6\gamma\epsilon\hbar} \gamma \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \frac{1}{\hbar} \\
 & 4 e^{-8\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] - \frac{1}{\hbar} \\
 & 4 e^{-6\gamma\epsilon\hbar} \gamma U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & \frac{3 \in U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} + \frac{e^{-3\gamma\epsilon\hbar} \in U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} + \\
 & \frac{5 e^{-2\gamma\epsilon\hbar} \in U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} + \frac{7 e^{-\gamma\epsilon\hbar} \in U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} + \\
 & \frac{U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} - \frac{e^{-4\gamma\epsilon\hbar} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} - \\
 & \frac{2 e^{-3\gamma\epsilon\hbar} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} +
 \end{aligned}$$

$$\begin{aligned}
& \frac{2 e^{-\gamma \in \hbar} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e]}{\hbar^2} - \frac{1}{\hbar^2} \\
& e^{-5 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2} \\
& 4 e^{-4 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2} \\
& 3 e^{-3 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] + \frac{1}{\hbar^2} \\
& e^{-6 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] + \frac{1}{\hbar^2} \\
& e^{-5 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2} \\
& e^{-4 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2} \\
& e^{-3 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \frac{1}{\hbar} \\
& e^{-5 \gamma \in \hbar} \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \frac{1}{\hbar} \\
& 2 e^{-4 \gamma \in \hbar} \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \frac{1}{\hbar} \\
& e^{-3 \gamma \in \hbar} \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \frac{1}{\hbar} \\
& 4 e^{-2 \gamma \in \hbar} \in U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \frac{1}{\hbar} \\
& e^{-4 \gamma \in \hbar} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \frac{1}{\hbar} \\
& e^{-2 \gamma \in \hbar} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \frac{1}{\hbar} \\
& e^{-9 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \frac{1}{\hbar} \\
& 2 e^{-8 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \frac{1}{\hbar} \\
& e^{-7 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \frac{1}{\hbar} \\
& 4 e^{-6 \gamma \in \hbar} \in U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \frac{1}{\hbar} \\
& e^{-8 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \frac{1}{\hbar} \\
& e^{-6 \gamma \in \hbar} U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, g, e, e]
\end{aligned}$$

The R-Matrix

Using Quesne's formula.

```

R_{i,j}_[d_] := Module[{x, y}, O[
  U_{-i}[x1 -> h, x2 -> f] U_i[] U_{-j}[] U_j[y1 -> g, y2 -> e],
  Series[Exp[h x1 y1 + \sum_{k=1}^d \frac{(1 - e^{\hbar \gamma \epsilon})^k (\hbar x_2 y_2)^k}{k (1 - e^{k \hbar \gamma \epsilon})}], {\hbar, \theta, d}]
]]

```

R_{1,2}[1]

$$U_{-2}[] U_{-1}[] U_1[] U_2[] + \hbar U_{-2}[] U_{-1}[f] U_1[] U_2[e] + \hbar U_{-2}[] U_{-1}[h] U_1[] U_2[g]$$

$$\text{\$TD} = 1; (R_{1,2}[1] R_{3,4}[1] // S_4 // m_{1,3 \rightarrow 1} // m_{2,4 \rightarrow 2}) /. U_{-4}[] U_{-3}[] \rightarrow 1$$

$$U_{-2}[] U_{-1}[] U_1[] U_2[] + \hbar U_{-2}[] U_{-1}[f] U_1[] U_2[e] - \hbar U_{-4}[] U_{-3}[h] U_{-2}[] U_{-1}[] U_1[] U_2[g] + \hbar U_{-2}[] U_{-1}[h] U_1[] U_2[g] - e^{\gamma \epsilon \hbar} \hbar U_{-4}[] U_{-3}[f] U_{-2}[] U_{-1}[] U_1[] U_2[G[-1], e]$$

$$\text{\$TD} = 1; S_{-1}[R_{1,2}[1]] ** R_{1,2}[1]$$

$$U_{-2}[] U_{-1}[] U_1[] U_2[] + \hbar U_{-2}[] U_{-1}[f] U_1[] U_2[e] - \hbar U_{-2}[] U_{-1}[H[-1], f] U_1[] U_2[e]$$

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