Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-05/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as $e^{\hbar h}$).
2. dm is not meta-associative.
3. R doesn't satisfy YB.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The "degree carrier" is $\hbar$, and all "coupling constants" are proportional to it.

```
$TD = 3; ℏ /: ℏ^d_. /; d > $TD := 0;
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\hbar \epsilon g}$ and to $H = e^{\hbar h}$, where $[g, e] = e$ and $[h, f] = -\epsilon f$.
Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2,\ G_1 G_2,\ e_1 + G_1 e_2,\ h_1 + h_2,\ H_1 H_2,\ f_1 H_2 + f_2)$.
Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;
$PBWRule = {G[_] → 1, g → 2, e → 3, H[_] → 4, h → 5, f → 6};
```

```
B[g, e] = e; B[e, G[n_]] = (e^(-n ℏ ϵ) - 1) U[G[n], e]; B[g, G[_]] = 0;
B[h, f] = -ϵ f; B[f, H[n_]] = (e^(n ℏ ϵ) - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on penseive://Projects/UEA/.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^p_] := U_i[x[n p]];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] ⧴ U_i[xs],
    x : AlgebraAtom ⧴ U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_{i_,j_→k_}[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_{i_,j_→k_}[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_{i_,j_→k_}[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_{i,j→k}];
m_{i_,j_→k_}[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_{i,j→i}) U_j[yy] // Expand // m_{i,j→k})
     c // USimp
  ];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] ⧴ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
   {Sx = Supp[x], Sy = Supp[y], is, σ, z},
   If[MatchQ[Sx ⋃ Sy, {_Integer ...}] && Min[Sx ⋃ Sy] < 0,
    is = Abs[Sx] ⋂ Abs[Sy];
    z = x; Do[z = m_{-i→-σ@i}[m_{i→σ@i}[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_{σ@i,i→i}[z], {i, is}];
    z,
    (* else *) is = Sx ⋂ Sy;
    z = x; Do[z = m_{i→σ@i}[z], {i, is}];
    z = USimp[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}];
    z
   ]
  ];
UB[x_, y_] := USimp[x ** y - y ** x];
```

```
𝕆[func_] := Normal@Series[func, {ℏ, 0, $TD}];
𝕆[specs_, func_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] ⧴ Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[𝕆[func], vars] /. (ps_ → c_) ⧴ c (
        specs /. MapThread[((#1 → _) ⧴ #3^#2) &, {vars, ps, elems}] /. U_i_ ⧴ UU_i
      )]
 ]
```

The 2D Lie BiAlgebra Pencil, Testing

```
𝕆[U₁[x → g], ℯ^(ℏ ϵ x)]
```

$U_1[] + \epsilon\hbar\, U_1[g] + \frac{1}{2}\epsilon^2\hbar^2\, U_1[g, g] + \frac{1}{6}\epsilon^3\hbar^3\, U_1[g, g, g]$

```
USimp@With[{G = 𝕆[U₁[x → g], ℯ^(ℏ ϵ x)]}, UB[U₁[e], G] - 𝕆[ℯ^(-ℏ ϵ) - 1] G ** U₁[e]]
```

0

```
B[e, G[3]]
```

$\left(-1 + e^{-3\epsilon\hbar}\right) U[G[3], e]$

```
$TD = 6;
USimp@With[{H = 𝕆[U₁[x → h], ℯ^(ℏ x)]}, UB[U₁[f], H] - 𝕆[ℯ^(ℏ ϵ) - 1] H ** U₁[f]]
```

0

```
x = U₁[g, G[2], e, e, e] U₂[g, g, e] U₃[g, g, G[-3], e];
(x // m_{1,2→1} // m_{1,3→1}) - (x // m_{2,3→2} // m_{1,2→1})
```

0

```
x = U₁[h, H[2], f, f, f] U₂[h, h, f] U₃[h, h, H[-3], f];
(x // m_{1,2→1} // m_{1,3→1}) - (x // m_{2,3→2} // m_{1,2→1})
```

0

The Co-Product and Co-Associativity

```
qΔ_{i_→j_,k_}[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
    U_i[] → U_j[] U_k[],
    U_i[g, xs___] :→
     (USimp[(U_j[g] U_k[] + U_j[] U_k[g]) qΔ_{i→tj,tk}[U_i[xs]]] // m_{j,tj→j} // m_{k,tk→k}),
    U_i[G[n_], xs___] :→ (USimp[U_j[G[n]] U_k[G[n]] qΔ_{i→tj,tk}[U_i[xs]]] //
       m_{j,tj→j} // m_{k,tk→k}),
    U_i[e, xs___] :→ (USimp[(U_j[e] U_k[G[1]] + U_j[] U_k[e]) qΔ_{i→tj,tk}[U_i[xs]]] //
       m_{j,tj→j} // m_{k,tk→k}),
    U_i[h, xs___] :→ (USimp[(U_j[h] U_k[] + U_j[] U_k[h]) qΔ_{i→tj,tk}[U_i[xs]]] //
       m_{j,tj→j} // m_{k,tk→k}),
    U_i[H[n_], xs___] :→ (USimp[U_j[H[n]] U_k[H[n]] qΔ_{i→tj,tk}[U_i[xs]]] //
       m_{j,tj→j} // m_{k,tk→k}),
    U_i[f, xs___] :→ (USimp[(U_j[f] U_k[] + U_j[H[1]] U_k[f]) qΔ_{i→tj,tk}[U_i[xs]]] //
       m_{j,tj→j} // m_{k,tk→k})
   }];
```

```
qΔ_{i_→j_,k_,l__}[ℰ_] := ℰ // qΔ_{i→j,k} // qΔ_{k→k,l}
```

```
U_1[e] // qΔ_{1→1,2}
```

```
U_1[] U_2[e] + U_1[e] U_2[G[1]]
```

```
{lhs = U_1[e] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U_1[e] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}
```

```
{U_1[] U_2[] U_3[e] + U_1[] U_2[e] U_3[G[1]] + U_1[e] U_2[G[1]] U_3[G[1]],
 U_1[] U_2[] U_3[e] + U_1[] U_2[e] U_3[G[1]] + U_1[e] U_2[G[1]] U_3[G[1]], True}
```

```
U_1[f] // qΔ_{1→1,2}
```

```
U_1[f] U_2[] + U_1[H[1]] U_2[f]
```

```
{lhs = U_1[f] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U_1[f] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}
```

```
{U_1[f] U_2[] U_3[] + U_1[H[1]] U_2[f] U_3[] + U_1[H[1]] U_2[H[1]] U_3[f],
 U_1[f] U_2[] U_3[] + U_1[H[1]] U_2[f] U_3[] + U_1[H[1]] U_2[H[1]] U_3[f], True}
```

```
x = U_1[g, G[2], e, e, e] U_2[g, g, G[-3], e];
(x // m_{1,2→1} // qΔ_{1→1,2}) - (x // qΔ_{2→3,4} // qΔ_{1→1,2} // m_{1,3→1} // m_{2,4→2})
```

```
0
```

```
x = U_1[h, H[2], f, f, f] U_2[h, h, H[-3], f];
(x // m_{1,2→1} // qΔ_{1→1,2}) - (x // qΔ_{2→3,4} // qΔ_{1→1,2} // m_{1,3→1} // m_{2,4→2})
```

```
0
```

The Antipode

Why o why this annoyance of left-vs-right?

```
S[g] = -g; S[G[n_]] := G[-n]; S[e] = -ⅇ^(ℏ ϵ) U[G[-1], e];
S[h] = -h; S[H[n_]] := H[-n]; S[f] = -U[H[-1], f];
S_i_[ℰ_] := Module[{ti}, USimp[
    ℰ /. U_i[x_, xs___] :→ m_{ti,i→i}[Expand[UU_i[S[x]] S_ti[U_ti[xs]]]]
  ]];
```

```
{lhs = S_1[U_1[e]], rhs = -U_1[e] ** U_1[G[-1]], lhs == rhs}
```

$\{-e^{\epsilon\hbar} U_1[G[-1], e], -e^{\epsilon\hbar} U_1[G[-1], e], \text{True}\}$

```
U_1[e] // S_1 // S_1
```

$e^{\epsilon\hbar} U_1[e]$

```
U_1[f] // S_1 // S_1
```

$e^{\epsilon\hbar} U_1[f]$

```
S_1[U_1[g, G[3], e, e]]
```

$2 e^{9\epsilon\hbar} U_1[G[-5], e, e] - e^{9\epsilon\hbar} U_1[G[-5], g, e, e]$

```
U_1[g, G[3], e, e] // qΔ_{1→1,2}
```

$U_1[G[3], g, e, e]\, U_2[G[5]] + U_1[G[3], g, e]\, U_2[G[4], e] + e^{-\epsilon\hbar} U_1[G[3], g, e]\, U_2[G[4], e] + U_1[G[3], e, e]\, U_2[G[5], g] + U_1[G[3], g]\, U_2[G[3], e, e] + U_1[G[3], e]\, U_2[G[4], g, e] + e^{-\epsilon\hbar} U_1[G[3], e]\, U_2[G[4], g, e] + U_1[G[3]]\, U_2[G[3], g, e, e]$

```
U_1[g, G[3], e, e] // qΔ_{1→1,2} // S_2
```

$U_1[G[3], g, e, e]\, U_2[G[-5]] - e^{4\epsilon\hbar} U_1[G[3], e]\, U_2[G[-5], e] - e^{5\epsilon\hbar} U_1[G[3], e]\, U_2[G[-5], e] - e^{4\epsilon\hbar} U_1[G[3], g, e]\, U_2[G[-5], e] - e^{5\epsilon\hbar} U_1[G[3], g, e]\, U_2[G[-5], e] - U_1[G[3], e, e]\, U_2[G[-5], g] + 2 e^{9\epsilon\hbar} U_1[G[3]]\, U_2[G[-5], e, e] + e^{9\epsilon\hbar} U_1[G[3], g]\, U_2[G[-5], e, e] + e^{4\epsilon\hbar} U_1[G[3], e]\, U_2[G[-5], g, e] + e^{5\epsilon\hbar} U_1[G[3], e]\, U_2[G[-5], g, e] - e^{9\epsilon\hbar} U_1[G[3]]\, U_2[G[-5], g, e, e]$

```
test = U_1[g, G[3], e, e];
{test // qΔ_{1→1,2} // S_2 // m_{1,2→1}, test // qΔ_{1→1,2} // S_2 // m_{2,1→1},
 test // qΔ_{1→1,2} // S_1 // m_{1,2→1}, test // qΔ_{1→1,2} // S_1 // m_{2,1→1}}
```

$\{0, 0, 0, 2 e^{6\epsilon\hbar} U_1[e, e] - 2 e^{7\epsilon\hbar} U_1[e, e] - 2 e^{8\epsilon\hbar} U_1[e, e] + 2 e^{9\epsilon\hbar} U_1[e, e]\}$

```
test = U_1[h, H[3], f, f];
{test // qΔ_{1→1,2} // S_2 // m_{1,2→1}, test // qΔ_{1→1,2} // S_2 // m_{2,1→1},
 test // qΔ_{1→1,2} // S_1 // m_{1,2→1}, test // qΔ_{1→1,2} // S_1 // m_{2,1→1}}
```

$\{0, 0, 0, -2 e^{-10\epsilon\hbar} \epsilon\, U_1[H[-2], f, f] + 2 e^{-9\epsilon\hbar} \epsilon\, U_1[H[-2], f, f] + 2 e^{-8\epsilon\hbar} \epsilon\, U_1[H[-2], f, f] - 2 e^{-7\epsilon\hbar} \epsilon\, U_1[H[-2], f, f]\}$

```
x = U_1[h, H[2], f, f, f] U_2[h, h, H[-3], f];
(x // m_{1,2→1} // S_1) - (x // S_1 // S_2 // m_{2,1→1})
```

0

```
x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
```

0

```
x = U1[];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

$U_1[]$

```
x = U1[];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

$U_1[]$

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

0

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

0

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

0

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

0

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[]] = 1;
P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
( P[U[g], U[h]] = ℏ^-1          P[U[g], U[H[n_]]] = n          P[U[g], U[f]] = 0
  P[U[G[n_]], U[h]] = n ϵ       P[U[G[n_]], U[H[m_]]] = e^(n m ℏ ϵ)   P[U[G[_]], U[f]] = 0
  P[U[e], U[h]] = 0             P[U[e], U[H[_]]] = 0           P[U[e], U[f]] = ℏ^-1 );
```

```
P_{i_,j_}[ℰ_] := USimp[ℰ /. U_i[xs___] U_j[ys___] → P[U[xs], U[ys]]];
```

```
t = Ui[g] Uj[e] Uk[f];
{mi,j→i[t] - mj,i→i[t], qΔk→k,l[t] - qΔk→l,k[t]}
```

$\{U_i[e]\, U_k[f],\ U_i[g]\, U_j[e]\, U_k[f]\, U_l[] - U_i[g]\, U_j[e]\, U_k[]\, U_l[f] + U_i[g]\, U_j[e]\, U_k[H[1]]\, U_l[f] - U_i[g]\, U_j[e]\, U_k[f]\, U_l[H[1]]\}$

```
t = U_i[g] U_j[e] U_k[f];
{(m_{i,j→i}[t] - m_{j,i→i}[t]) // P_{i,k}, (qΔ_{k→k,l}[t] - qΔ_{k→l,k}[t]) // P_{i,k} // P_{j,l}}
```

$\left\{\frac{1}{\hbar}, \frac{1}{\hbar}\right\}$

```
Table[t = U_i[xi] U_j[xj] U_k[yk];
 {(m_{i,j→i}[t] - m_{j,i→i}[t]) // P_{i,k}, (qΔ_{k→k,l}[t] - qΔ_{k→l,k}[t]) // P_{i,k} // P_{j,l}},
 {xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
```

$\left\{\left\{\{\{0,0\},\{0,0\}\},\left\{\{0,0\},\left\{\frac{1}{\hbar},\frac{1}{\hbar}\right\}\right\}\right\},\left\{\left\{\{0,0\},\left\{-\frac{1}{\hbar},-\frac{1}{\hbar}\right\}\right\},\{\{0,0\},\{0,0\}\}\right\}\right\}$

```
Table[t = U_i[xi] U_k[yk] U_l[yl];
 {(qΔ_{i→i,j}[t] - qΔ_{i→j,i}[t]) // P_{i,k} // P_{j,l}, (m_{k,l→k}[t] - m_{l,k→k}[t]) // P_{i,k}},
 {xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
```

$\left\{\left\{\{\{0,0\},\{0,0\}\},\{\{0,0\},\{0,0\}\}\right\},\left\{\left\{\{0,0\},\left\{-\frac{\epsilon}{\hbar},-\frac{\epsilon}{\hbar}\right\}\right\},\left\{\left\{\frac{\epsilon}{\hbar},\frac{\epsilon}{\hbar}\right\},\{0,0\}\right\}\right\}\right\}$

General Pairings

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[x_, xs__], U[y_]] := P[U[x, xs], U[y]] =
   Module[{i, j, k, l}, USimp[U_i[x] U_j[xs] qΔ_{k→k,l}[U_k[y]]] // P_{i,k} // P_{j,l}];
P[U[xs__], U[y_, ys__]] := P[U[xs], U[y, ys]] =
   Module[{i, j, k, l}, USimp[qΔ_{i→i,j}[U_i[xs]] U_k[y] U_l[ys]] // P_{i,k} // P_{j,l}];
```

```
{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
```

$\left\{0, \frac{1}{\hbar}, 0\right\}$

```
P[U[e], U[f, f]]
```

0

```
P[U[e, e], U[f, f]]
```

$\frac{1}{\hbar^2} + \frac{e^{\epsilon\hbar}}{\hbar^2}$

```
lhs = Factor@Table[ħ^n P[U@@Table[e, {n}], U@@Table[f, {n}]], {n, $TD = 7}]
```

$\{1, 1+e^{\epsilon\hbar}, (1+e^{\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}), (1+e^{\epsilon\hbar})^2(1+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}),$
$(1+e^{\epsilon\hbar})^2(1+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}),$
$(1+e^{\epsilon\hbar})^3(1+e^{2\epsilon\hbar})(1-e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})^2(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}),$
$(1+e^{\epsilon\hbar})^3(1+e^{2\epsilon\hbar})(1-e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})^2$
$(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}+e^{5\epsilon\hbar}+e^{6\epsilon\hbar})\}$

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^(ħ ϵ)], {n, $TD = 7}]
```

$$\{1,\ 1+e^{\epsilon\hbar},\ (1+e^{\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}),\ (1+e^{\epsilon\hbar})^2(1+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}),$$
$$(1+e^{\epsilon\hbar})^2(1+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}),$$
$$(1+e^{\epsilon\hbar})^3(1+e^{2\epsilon\hbar})(1-e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})^2(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}),$$
$$(1+e^{\epsilon\hbar})^3(1+e^{2\epsilon\hbar})(1-e^{\epsilon\hbar}+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})^2$$
$$(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar}+e^{3\epsilon\hbar}+e^{4\epsilon\hbar}+e^{5\epsilon\hbar}+e^{6\epsilon\hbar})\}$$

```
MapThread[Equal, {lhs, rhs}]
```

{True, True, True, True, True, True, True}

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

$$\frac{120}{\hbar^5}$$

```
P[U[g, g, g, g, g, e, e, e, e], U[h, h, h, h, h, f, f, f, f]] // Factor
```

$$\frac{120\,(1+e^{\epsilon\hbar})^2(1+e^{2\epsilon\hbar})(1+e^{\epsilon\hbar}+e^{2\epsilon\hbar})}{\hbar^9}$$

```
x = U1[g, G[-3], e, e] U2[g, g, G[1], e] U3[h, H[2], f, f, f];
(x // m1,2→1 // P1,3) - (x // qΔ3→3,4 // P1,3 // P2,4)
```

0

```
x = U1[h, H[-3], f, f] U2[h, h, H[1], f] U3[g, G[2], e, e, e];
(x // m1,2→1 // P3,1) - (x // qΔ3→3,4 // P3,1 // P4,2)
```

0

```
x = U1[g, g, G[-3], e, e, e] U2[h, h, H[2], f, f, f];
(x // S1 // P1,2) - (x // S2 // P1,2)
```

0

The Double

```
dm[i_,j_→k_][ℰ_] := Module[{t1, t2, t3, h1, h2, h3},
  ℰ // qΔ[i→h1,h2,h3] // S[h1] // qΔ[-j→t1,t2,t3] // P[h1,t1] // P[h3,t3] // m[h2,j→k] // m[-i,t2→-k]]
```

```
U-1[] U1[g] U-2[h] U2[] // dm1,2→1
```

$U_{-1}[h]\, U_1[g]$

```
U-1[] U1[g] U-2[f] U2[] // dm1,2→1
```

$-U_{-1}[f]\, U_1[] + U_{-1}[f]\, U_1[g]$

```
U-1[] U1[G[1]] U-2[f] U2[] // dm1,2→1
```

$e^{-\epsilon\hbar}\, U_{-1}[f]\, U_1[G[1]]$

$U_{-1}[]\, U_1[e]\, U_{-2}[h]\, U_2[]$ // $dm_{1,2\to1}$

$\epsilon\, U_{-1}[]\, U_1[e] + U_{-1}[h]\, U_1[e]$

$U_{-1}[]\, U_1[e]\, U_{-2}[H[1]]\, U_2[]$ // $dm_{1,2\to1}$

$e^{\epsilon\hbar}\, U_{-1}[H[1]]\, U_1[e]$

$U_{-1}[]\, U_1[e]\, U_{-2}[f]\, U_2[]$ // $dm_{1,2\to1}$

$$\frac{U_{-1}[H[1]]\, U_1[]}{\hbar} + U_{-1}[f]\, U_1[e] - \frac{U_{-1}[]\, U_1[G[1]]}{\hbar}$$

x = $U_{-1}[]\, U_1[e]\, U_{-2}[f]\, U_2[]\, U_{-3}[h]\, U_3[]$;
(x // $dm_{1,2\to1}$ // $dm_{1,3\to1}$) - (x // $dm_{2,3\to2}$ // $dm_{1,2\to1}$)

$$-\frac{\epsilon\, U_{-1}[H[1]]\, U_1[]}{\hbar} + \frac{\epsilon\, U_{-1}[]\, U_1[G[1]]}{\hbar}$$

x = $U_{-1}[h, h, f]\, U_1[G[2], e, e]\, U_{-2}[H[-1], f, f]\, U_2[g]\, U_{-3}[h, f]\, U_3[g, g, e]$;
(x // $dm_{1,2\to1}$ // $dm_{1,3\to1}$) - (x // $dm_{2,3\to2}$ // $dm_{1,2\to1}$)

$$\frac{2 e^{-5\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} - \frac{4 e^{-4\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} +$$
$$\frac{2 e^{-3\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} + \frac{8 e^{-2\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} -$$
$$\frac{2 e^{-4\epsilon\hbar} U_{-1}[h,h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} + \frac{2 e^{-2\epsilon\hbar} U_{-1}[h,h,h,f,f,f]\, U_1[G[2],e,e]}{\hbar} -$$
$$\frac{1}{\hbar} 2 e^{-9\epsilon\hbar}\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],e,e] + \frac{1}{\hbar}$$
$$4 e^{-8\epsilon\hbar}\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],e,e] - \frac{1}{\hbar}$$
$$2 e^{-7\epsilon\hbar}\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],e,e] - \frac{1}{\hbar}$$
$$8 e^{-6\epsilon\hbar}\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],e,e] + \frac{1}{\hbar}$$
$$2 e^{-8\epsilon\hbar} U_{-1}[H[-1],h,h,h,f,f,f]\, U_1[G[3],e,e] - \frac{1}{\hbar}$$
$$2 e^{-6\epsilon\hbar} U_{-1}[H[-1],h,h,h,f,f,f]\, U_1[G[3],e,e] -$$
$$\frac{5 e^{-5\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} + \frac{10 e^{-4\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} -$$
$$\frac{5 e^{-3\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} - \frac{20 e^{-2\epsilon\hbar}\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} +$$
$$\frac{5 e^{-4\epsilon\hbar} U_{-1}[h,h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} - \frac{5 e^{-2\epsilon\hbar} U_{-1}[h,h,h,f,f,f]\, U_1[G[2],g,e,e]}{\hbar} +$$
$$\frac{\epsilon\, U_{-1}[H[1],h,h,f,f]\, U_1[G[2],g,g,e]}{\hbar^2} + \frac{4 e^{\epsilon\hbar}\epsilon\, U_{-1}[H[1],h,h,f,f]\, U_1[G[2],g,g,e]}{\hbar^2} +$$
$$\frac{1}{\hbar^2} 3 e^{2\epsilon\hbar}\epsilon\, U_{-1}[H[1],h,h,f,f]\, U_1[G[2],g,g,e] - \frac{U_{-1}[H[1],h,h,h,f,f]\, U_1[G[2],g,g,e]}{\hbar^2} -$$
$$\frac{e^{-\epsilon\hbar} U_{-1}[H[1],h,h,h,f,f]\, U_1[G[2],g,g,e]}{\hbar^2} + \frac{e^{\epsilon\hbar} U_{-1}[H[1],h,h,h,f,f]\, U_1[G[2],g,g,e]}{\hbar^2} +$$

$$\frac{e^{2\epsilon\hbar}\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,e]}{\hbar^2}+\frac{1}{\hbar}$$
$$5\,e^{-9\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]-\frac{1}{\hbar}$$
$$10\,e^{-8\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]+\frac{1}{\hbar}$$
$$5\,e^{-7\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]+\frac{1}{\hbar}$$
$$20\,e^{-6\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]-\frac{1}{\hbar}$$
$$5\,e^{-8\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,e,e]+\frac{1}{\hbar}$$
$$5\,e^{-6\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,e,e]-\frac{3\,\epsilon\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}-$$
$$\frac{e^{-3\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}-\frac{5\,e^{-2\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}-$$
$$\frac{7\,e^{-\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}-\frac{U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}+$$
$$\frac{e^{-4\epsilon\hbar}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}+\frac{2\,e^{-3\epsilon\hbar}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}-$$
$$\frac{2\,e^{-\epsilon\hbar}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]}{\hbar^2}+\frac{1}{\hbar^2}e^{-5\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]+$$
$$\frac{1}{\hbar^2}4\,e^{-4\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]+\frac{1}{\hbar^2}$$
$$3\,e^{-3\epsilon\hbar}\,\epsilon\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]-\frac{1}{\hbar^2}$$
$$e^{-6\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]-\frac{1}{\hbar^2}$$
$$e^{-5\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\frac{1}{\hbar^2}$$
$$e^{-4\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\frac{1}{\hbar^2}$$
$$e^{-3\epsilon\hbar}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\frac{1}{\hbar}$$
$$4\,e^{-5\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\frac{1}{\hbar}$$
$$8\,e^{-4\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\frac{1}{\hbar}$$
$$4\,e^{-3\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\frac{1}{\hbar}$$
$$16\,e^{-2\epsilon\hbar}\,\epsilon\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\frac{1}{\hbar}$$
$$4\,e^{-4\epsilon\hbar}\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\frac{1}{\hbar}$$
$$4\,e^{-2\epsilon\hbar}\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\frac{\epsilon\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,g,e]}{\hbar^2}-$$

$$\frac{1}{\hbar^2} 4\, e^{\epsilon \hbar} \epsilon\, U_{-1}[H[1], h, h, f, f]\, U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2}$$
$$3\, e^{2 \epsilon \hbar} \epsilon\, U_{-1}[H[1], h, h, f, f]\, U_1[G[2], g, g, g, e] +$$
$$\frac{U_{-1}[H[1], h, h, h, f, f]\, U_1[G[2], g, g, g, e]}{\hbar^2} + \frac{1}{\hbar^2}$$
$$e^{-\epsilon \hbar}\, U_{-1}[H[1], h, h, h, f, f]\, U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2}$$
$$e^{\epsilon \hbar}\, U_{-1}[H[1], h, h, h, f, f]\, U_1[G[2], g, g, g, e] - \frac{1}{\hbar^2}$$
$$e^{2 \epsilon \hbar}\, U_{-1}[H[1], h, h, h, f, f]\, U_1[G[2], g, g, g, e] - \frac{1}{\hbar}$$
$$4\, e^{-9 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f, f]\, U_1[G[3], g, g, e, e] + \frac{1}{\hbar}$$
$$8\, e^{-8 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f, f]\, U_1[G[3], g, g, e, e] - \frac{1}{\hbar}$$
$$4\, e^{-7 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f, f]\, U_1[G[3], g, g, e, e] - \frac{1}{\hbar}$$
$$16\, e^{-6 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f, f]\, U_1[G[3], g, g, e, e] + \frac{1}{\hbar}$$
$$4\, e^{-8 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f, f]\, U_1[G[3], g, g, e, e] - \frac{1}{\hbar}$$
$$4\, e^{-6 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f, f]\, U_1[G[3], g, g, e, e] +$$
$$\frac{3 \epsilon\, U_{-1}[h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} + \frac{e^{-3 \epsilon \hbar} \epsilon\, U_{-1}[h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} +$$
$$\frac{5\, e^{-2 \epsilon \hbar} \epsilon\, U_{-1}[h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} + \frac{7\, e^{-\epsilon \hbar} \epsilon\, U_{-1}[h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} +$$
$$\frac{U_{-1}[h, h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} - \frac{e^{-4 \epsilon \hbar}\, U_{-1}[h, h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} -$$
$$\frac{2\, e^{-3 \epsilon \hbar}\, U_{-1}[h, h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} + \frac{2\, e^{-\epsilon \hbar}\, U_{-1}[h, h, h, f, f]\, U_1[G[3], g, g, g, e]}{\hbar^2} -$$
$$\frac{1}{\hbar^2} e^{-5 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f]\, U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2}$$
$$4\, e^{-4 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f]\, U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2}$$
$$3\, e^{-3 \epsilon \hbar} \epsilon\, U_{-1}[H[-1], h, h, f, f]\, U_1[G[4], g, g, g, e] + \frac{1}{\hbar^2}$$
$$e^{-6 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f]\, U_1[G[4], g, g, g, e] + \frac{1}{\hbar^2}$$
$$e^{-5 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f]\, U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2}$$
$$e^{-4 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f]\, U_1[G[4], g, g, g, e] - \frac{1}{\hbar^2}$$
$$e^{-3 \epsilon \hbar}\, U_{-1}[H[-1], h, h, h, f, f]\, U_1[G[4], g, g, g, e] - \frac{1}{\hbar}$$
$$e^{-5 \epsilon \hbar} \epsilon\, U_{-1}[h, h, f, f, f]\, U_1[G[2], g, g, g, e, e] + \frac{1}{\hbar}$$

$$2\, e^{-4\epsilon\hbar}\,\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,g,g,e,e] - \frac{1}{\hbar}$$
$$e^{-3\epsilon\hbar}\,\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,g,g,e,e] - \frac{1}{\hbar}$$
$$4\, e^{-2\epsilon\hbar}\,\epsilon\, U_{-1}[h,h,f,f,f]\, U_1[G[2],g,g,g,e,e] + \frac{1}{\hbar}$$
$$e^{-4\epsilon\hbar}\, U_{-1}[h,h,h,f,f,f]\, U_1[G[2],g,g,g,e,e] - \frac{1}{\hbar}$$
$$e^{-2\epsilon\hbar}\, U_{-1}[h,h,h,f,f,f]\, U_1[G[2],g,g,g,e,e] + \frac{1}{\hbar}$$
$$e^{-9\epsilon\hbar}\,\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],g,g,g,e,e] - \frac{1}{\hbar}$$
$$2\, e^{-8\epsilon\hbar}\,\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],g,g,g,e,e] + \frac{1}{\hbar}$$
$$e^{-7\epsilon\hbar}\,\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],g,g,g,e,e] + \frac{1}{\hbar}$$
$$4\, e^{-6\epsilon\hbar}\,\epsilon\, U_{-1}[H[-1],h,h,f,f,f]\, U_1[G[3],g,g,g,e,e] - \frac{1}{\hbar}$$
$$e^{-8\epsilon\hbar}\, U_{-1}[H[-1],h,h,h,f,f,f]\, U_1[G[3],g,g,g,e,e] + \frac{1}{\hbar}$$
$$e^{-6\epsilon\hbar}\, U_{-1}[H[-1],h,h,h,f,f,f]\, U_1[G[3],g,g,g,e,e]$$

The R-Matrix

Using Quesne’s formula.

```
R_{i_,j_}[d_] := Module[{x, y}, 𝕆[
   U_{-i}[x1 → h, x2 → f] U_i[] U_{-j}[] U_j[y1 → g, y2 → e],
   Series[Exp[ħ x1 y1 + Sum[(1 - e^(ħ ϵ))^k (ħ x2 y2)^k / (k (1 - e^(k ħ ϵ))), {k, 1, d}]], {ħ, 0, d}]
  ]]
```

```
R_{1,2}[1]
```

$$U_{-2}[]\, U_{-1}[]\, U_1[]\, U_2[] + \hbar\, U_{-2}[]\, U_{-1}[f]\, U_1[]\, U_2[e] + \hbar\, U_{-2}[]\, U_{-1}[h]\, U_1[]\, U_2[g]$$

```
$TD = 1; (R_{1,2}[1] R_{3,4}[1] // S_4 // m_{1,3→1} // m_{2,4→2}) /. U_{-4}[] U_{-3}[] -> 1
```

$$U_{-2}[]\, U_{-1}[]\, U_1[]\, U_2[] + \hbar\, U_{-2}[]\, U_{-1}[f]\, U_1[]\, U_2[e] - \hbar\, U_{-4}[]\, U_{-3}[h]\, U_{-2}[]\, U_{-1}[]\, U_1[]\, U_2[g] + \hbar\, U_{-2}[]\, U_{-1}[h]\, U_1[]\, U_2[g] - e^{\epsilon\hbar}\,\hbar\, U_{-4}[]\, U_{-3}[f]\, U_{-2}[]\, U_{-1}[]\, U_1[]\, U_2[G[-1],e]$$

```
$TD = 1; S_{-1}[R_{1,2}[1]] ** R_{1,2}[1]
```

$$U_{-2}[]\, U_{-1}[]\, U_1[]\, U_2[] + \hbar\, U_{-2}[]\, U_{-1}[f]\, U_1[]\, U_2[e] - \hbar\, U_{-2}[]\, U_{-1}[H[-1],f]\, U_1[]\, U_2[e]$$

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