

Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-05/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as $e^{\hbar h}$).
2. dm is not meta-associative.
3. R doesn't satisfy YB.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The “degree carrier” is \hbar , and all “coupling constants” are proportional to it.

```
$TD = 3;  $\hbar$  /:  $\hbar^{d-}$  /;  $d > $TD := 0$ ;
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\hbar \epsilon g}$ and to $H = e^{\hbar h}$, where $[g, e] = \hbar e$ and $[h, f] = -\hbar f$.

Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2, G_1 G_2, e_1 + G_1 e_2, h_1 + h_2, H_1 H_2, f_1 H_2 + f_2)$.

Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;
$PBWRule = {G[_] → 1, g → 2, e → 3, H[_] → 4, h → 5, f → 6};
```

```
B[g, e] =  $\hbar$  e; B[e, G[n_]] = (e^{-n $\hbar^2$  e} - 1) U[G[n], e]; B[g, G[_]] = 0;
B[h, f] = - $\hbar$  f; B[f, H[n_]] = (e^{n $\hbar^2$  e} - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on pensieve://Projects/UEA/.

```
B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```

x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^-] := U_i[x[n p]];
UU_i_[x^-] := UU_i@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];

```

```

m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_i->j_[ε_] := ε /. U_i -> U_j;

```

```

m_i->j->k_[c_. U_i[x_] U_j[]] := c U_k[x];
m_i->j->k_[c_. U_i[] U_j[y_]] := c U_k[y];
m_i->j->k_[c_. U_i[xx_, x_[n1_]] U_j[x_[n2_], yy_]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j->k];
m_i->j->k_[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];

```

```

Supp[ε_] := Union@Cases[{ε}, U_i[___] -> i, ∞];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
  {Sx = Supp[x], Sy = Supp[y], is, σ, z},
  If[MatchQ[Sx ∪ Sy, {_Integer ...}] && Min[Sx ∪ Sy] < 0,
    is = Abs[Sx] ∩ Abs[Sy];
    z = x; Do[z = m_i->σ@i[m_i->σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_σ@i,i->i[z], {i, is}];
    z,
    (* else *) is = Sx ∩ Sy;
    z = x; Do[z = m_i->σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = m_σ@i,i->i[z], {i, is}];
    z
  ]
];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[func_] := Normal@Series[func, {h, 0, $TD}];
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U[u___] => Cases[{u}, r_Rule], infinity];
  vars = First/@rules; elems = Last/@rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ -> c_) => c (
    specs /. MapThread[{(#1 -> _) => #3^#2} &, {vars, ps, elems}] /. U_i_ => UU_i
  )]
]

```

The 2D Lie BiAlgebra Pencil, Testing

$O[U_1[x \rightarrow g], e^{\hbar \epsilon x}]$

$$U_1[] + \epsilon \hbar U_1[g] + \frac{1}{2} \epsilon^2 \hbar^2 U_1[g, g] + \frac{1}{6} \epsilon^3 \hbar^3 U_1[g, g, g]$$

$USimp@With[{G = O[U_1[x \rightarrow g], e^{\hbar \epsilon x}]}, UB[U_1[e], G] - O[e^{-\hbar^2 \epsilon} - 1] G ** U_1[e]}$

0

$B[e, G[3]]$

$$(-1 + e^{-3 \epsilon \hbar^2}) U[G[3], e]$$

$\$TD = 6;$

$USimp@With[{H = O[U_1[x \rightarrow h], e^{\hbar x}]}, UB[U_1[f], H] - O[e^{\hbar^2 \epsilon} - 1] H ** U_1[f]}$

0

$x = U_1[g, G[2], e, e, e] U_2[g, g, e] U_3[g, g, G[-3], e];$

$$(x // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) - (x // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})$$

0

$x = U_1[h, H[2], f, f, f] U_2[h, h, f] U_3[h, h, H[-3], f];$

$$(x // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) - (x // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})$$

0

The Co-Product and Co-Associativity

```

qΔi→j,k[ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[g, xS____] =>
    (USimp[(Uj[g] Uk[] + Uj[] Uk[g]) qΔi→tj,tk[Ui[xS]]] // mj,tj→j // mk,tk→k),
  Ui[G[n-], xS____] => (USimp[Uj[G[n]] Uk[G[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[e, xS____] => (USimp[(Uj[e] Uk[G[1]] + Uj[] Uk[e]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[h, xS____] => (USimp[(Uj[h] Uk[] + Uj[] Uk[h]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[H[n-], xS____] => (USimp[Uj[H[n]] Uk[H[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[f, xS____] => (USimp[(Uj[f] Uk[] + Uj[H[1]] Uk[f]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k)
}];

```

```

qΔi→j,k,l[ε-] := ε // qΔi→j,k // qΔk→l

```

U₁[e] // qΔ_{1→1,2}

U₁[] U₂[e] + U₁[e] U₂[G[1]]

{lhs = U₁[e] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U₁[e] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}

{U₁[] U₂[] U₃[e] + U₁[] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]],
 U₁[] U₂[] U₃[e] + U₁[] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]], True}

U₁[f] // qΔ_{1→1,2}

U₁[f] U₂[] + U₁[H[1]] U₂[f]

{lhs = U₁[f] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U₁[f] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}

{U₁[f] U₂[] U₃[] + U₁[H[1]] U₂[f] U₃[] + U₁[H[1]] U₂[H[1]] U₃[f],
 U₁[f] U₂[] U₃[] + U₁[H[1]] U₂[f] U₃[] + U₁[H[1]] U₂[H[1]] U₃[f], True}

x = U₁[g, G[2], e, e, e] U₂[g, g, G[-3], e];

(x // m_{1,2→1} // qΔ_{1→1,2}) - (x // qΔ_{2→3,4} // qΔ_{1→1,2} // m_{1,3→1} // m_{2,4→2})

0

x = U₁[h, H[2], f, f, f] U₂[h, h, H[-3], f];

(x // m_{1,2→1} // qΔ_{1→1,2}) - (x // qΔ_{2→3,4} // qΔ_{1→1,2} // m_{1,3→1} // m_{2,4→2})

0

The Antipode

Why o why this annoyance of left-vs-right?

```

S[g] = -g; S[G[n_]] := G[-n]; S[e] = -eħ2 U[G[-1], e];
S[h] = -h; S[H[n_]] := H[-n]; S[f] = -U[H[-1], f];
Si[ε_] := Module[{ti}, USimp[
  ε /. Ui[X_, xS_____] => mti,i→i[Expand[UUi[S[X]] Sti[Uti[xS]]]]
]];

```

```
{lhs = S1[U1[e]], rhs = -U1[e] ** U1[G[-1]], lhs == rhs}
```

```
{-eħ2 U1[G[-1], e], -eħ2 U1[G[-1], e], True}
```

```
U1[e] // S1 // S1
```

```
eħ2 U1[e]
```

```
U1[f] // S1 // S1
```

```
eħ2 U1[f]
```

```
S1[U1[g, G[3], e, e]]
```

```
2 e9 ħ2 ħ U1[G[-5], e, e] - e9 ħ2 U1[G[-5], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2
```

```
U1[G[3], g, e, e] U2[G[5]] + U1[G[3], g, e] U2[G[4], e] +
e-ħ2 U1[G[3], g, e] U2[G[4], e] + U1[G[3], e, e] U2[G[5], g] + U1[G[3], g] U2[G[3], e, e] +
U1[G[3], e] U2[G[4], g, e] + e-ħ2 U1[G[3], e] U2[G[4], g, e] + U1[G[3]] U2[G[3], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2 // S2
```

```
U1[G[3], g, e, e] U2[G[-5]] -
e4 ħ2 ħ U1[G[3], e] U2[G[-5], e] - e5 ħ2 ħ U1[G[3], e] U2[G[-5], e] -
e4 ħ2 U1[G[3], g, e] U2[G[-5], e] - e5 ħ2 U1[G[3], g, e] U2[G[-5], e] -
U1[G[3], e, e] U2[G[-5], g] + 2 e9 ħ2 ħ U1[G[3]] U2[G[-5], e, e] +
e9 ħ2 U1[G[3], g] U2[G[-5], e, e] + e4 ħ2 U1[G[3], e] U2[G[-5], g, e] +
e5 ħ2 U1[G[3], e] U2[G[-5], g, e] - e9 ħ2 U1[G[3]] U2[G[-5], g, e, e]
```

```
test = U1[g, G[3], e, e];
```

```
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, 2 e6 ħ2 ħ U1[e, e] - 2 e7 ħ2 ħ U1[e, e] - 2 e8 ħ2 ħ U1[e, e] + 2 e9 ħ2 ħ U1[e, e]}
```

```
test = U1[h, H[3], f, f];
```

```
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, -2 e-10 ħ2 ħ U1[H[-2], f, f] + 2 e-9 ħ2 ħ U1[H[-2], f, f] +
2 e-8 ħ2 ħ U1[H[-2], f, f] - 2 e-7 ħ2 ħ U1[H[-2], f, f]}
```

```
x = U1[h, H[2], f, f, f] U2[h, h, H[-3], f];
```

```
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
```

```
0
```

```
x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
0
```

```
x = U1[];
(x // qΔ1→1,2 // S1 // m1,2→1)
U1[]
```

```
x = U1[];
(x // qΔ1→1,2 // S2 // m1,2→1)
U1[]
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

The Pairing at Lie-Level and Compatibilities

$$\begin{aligned}
 &P[U[], U[]] = 1; \\
 &P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1; \\
 &P[U[], U[[_]]] = P[U[[_]], U[]] = 0; \\
 &\left(\begin{array}{lll}
 P[U[g], U[h]] = 1 & P[U[g], U[H[n_]]] = n \hbar & P[U[g], U[f]] = 0 \\
 P[U[G[n_]], U[h]] = n \hbar \epsilon & P[U[G[n_]], U[H[m_]]] = e^{n m \hbar^2 \epsilon} & P[U[G[_]], U[f]] = 0 \\
 P[U[e], U[h]] = 0 & P[U[e], U[H[_]]] = 0 & P[U[e], U[f]] = 1
 \end{array} \right);
 \end{aligned}$$

$$P_{i,j}[\mathcal{E}] := \text{USimp}[\mathcal{E} /. U_i[xs_] U_j[ys_] \rightarrow P[U[xs], U[ys]]];$$

```
t = U_i[g] U_j[e] U_k[f];
{m_{j,i}[t] - m_{j,i}[t], qΔ_{k→k,1}[t] - qΔ_{k→1,k}[t]}
{ħ U_i[e] U_k[f], U_i[g] U_j[e] U_k[f] U_1[] - U_i[g] U_j[e] U_k[] U_1[f] +
 U_i[g] U_j[e] U_k[H[1]] U_1[f] - U_i[g] U_j[e] U_k[f] U_1[H[1]]}
```

```
t = Ui[g] Uj[e] Uk[f];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1}
{ħ, ħ}
```

```
Table[t = Ui[xi] Uj[xj] Uk[yk];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1},
{xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {ħ, ħ}}, {{{0, 0}, {-ħ, -ħ}}, {{0, 0}, {0, 0}}}}
```

```
Table[t = Ui[xi] Uk[yk] Ul[yl];
{(qΔi→i,j[t] - qΔi→j,i[t]) // Pi,k // Pj,1, (mk,1→k[t] - m1,k→k[t]) // Pi,k},
{xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{{0, 0}, {-εħ, -εħ}}, {{εħ, εħ}, {0, 0}}}}
```

General Pairings

The pairing sequence: (one,one) (above), (many,one), (many,many).

```
P[U[x_, xs_], U[y_]] := P[U[x, xs], U[y]] =
Module[{i, j, k, l}, USimp[Ui[x] Uj[xs] qΔk→k,1[Uk[y]]] // Pi,k // Pj,1];
P[U[xs_], U[y_, ys_]] := P[U[xs], U[y, ys]] =
Module[{i, j, k, l}, USimp[qΔi→i,j[Ui[xs]] Uk[y] Ul[ys]] // Pi,k // Pj,1];
```

```
{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
{0, ħ, 0}
```

```
P[U[e], U[f, f]]
```

```
0
```

```
P[U[e, e], U[f, f]]
```

```
1 + eεħ2
```

```
lhs = Factor@Table[P[U@@Table[e, {n}], U@@Table[f, {n}]], {n, 7}]
```

```
{1, 1 + eεħ2, (1 + eεħ2) (1 + eεħ2 + e2εħ2), (1 + eεħ2)2 (1 + e2εħ2) (1 + eεħ2 + e2εħ2),
(1 + eεħ2)2 (1 + e2εħ2) (1 + eεħ2 + e2εħ2) (1 + eεħ2 + e2εħ2 + e3εħ2 + e4εħ2),
(1 + eεħ2)3 (1 + e2εħ2) (1 - eεħ2 + e2εħ2) (1 + eεħ2 + e2εħ2)2 (1 + eεħ2 + e2εħ2 + e3εħ2 + e4εħ2),
(1 + eεħ2)3 (1 + e2εħ2) (1 - eεħ2 + e2εħ2) (1 + eεħ2 + e2εħ2)2
(1 + eεħ2 + e2εħ2 + e3εħ2 + e4εħ2) (1 + eεħ2 + e2εħ2 + e3εħ2 + e4εħ2 + e5εħ2 + e6εħ2)}
```

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^h^2 e], {n, 7}]
{1, 1 + e^h^2, (1 + e^h^2) (1 + e^h^2 + e^2 e^h^2), (1 + e^h^2)^2 (1 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2),
(1 + e^h^2)^2 (1 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2 + e^3 e^h^2 + e^4 e^h^2),
(1 + e^h^2)^3 (1 + e^2 e^h^2) (1 - e^h^2 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2)^2 (1 + e^h^2 + e^2 e^h^2 + e^3 e^h^2 + e^4 e^h^2),
(1 + e^h^2)^3 (1 + e^2 e^h^2) (1 - e^h^2 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2)^2
(1 + e^h^2 + e^2 e^h^2 + e^3 e^h^2 + e^4 e^h^2) (1 + e^h^2 + e^2 e^h^2 + e^3 e^h^2 + e^4 e^h^2 + e^5 e^h^2 + e^6 e^h^2)}
```

lhs == rhs

True

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

120

```
P[U[g, g, g, g, g, e, e, e, e], U[h, h, h, h, h, f, f, f, f]] // Factor
```

120 (1 + e^h^2)^2 (1 + e^2 e^h^2) (1 + e^h^2 + e^2 e^h^2)

```
x = U1[g, G[-3], e, e] U2[g, g, G[1], e] U3[h, H[2], f, f, f];
```

```
(x // m1,2→1 // P1,3) - (x // qΔ3→3,4 // P1,3 // P2,4)
```

0

```
x = U1[h, H[-3], f, f] U2[h, h, H[1], f] U3[g, G[2], e, e, e];
```

```
(x // m1,2→1 // P3,1) - (x // qΔ3→3,4 // P3,1 // P4,2)
```

0

```
x = U1[g, g, G[-3], e, e, e] U2[h, h, H[2], f, f, f];
```

```
(x // S1 // P1,2) - (x // S2 // P1,2)
```

0

The Double

```
dmi,j→k[E_] := Module[{t1, t2, t3, h1, h2, h3},
  E // qΔi→h1,h2,h3 // Sh1 // qΔj→t1,t2,t3 // Ph1,t1 // Ph3,t3 // mh2,j→k // m-i,t2→-k]
```

```
U-1[[]] U1[g] U-2[h] U2[[]] // dm1,2→1
```

U₋₁[h] U₁[g]

```
U-1[[]] U1[g] U-2[f] U2[[]] // dm1,2→1
```

-ħ U₋₁[f] U₁[[]] + U₋₁[f] U₁[g]

```
U-1[[]] U1[G[1]] U-2[f] U2[[]] // dm1,2→1
```

e^{-h^2} U₋₁[f] U₁[G[1]]

$$U_{-1}[] U_1[e] U_{-2}[h] U_2[] // dm_{1,2 \rightarrow 1}$$

$$\in \hbar U_{-1}[] U_1[e] + U_{-1}[h] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[H[1]] U_2[] // dm_{1,2 \rightarrow 1}$$

$$e^{\epsilon \hbar^2} U_{-1}[H[1]] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[f] U_2[] // dm_{1,2 \rightarrow 1}$$

$$U_{-1}[H[1]] U_1[] + U_{-1}[f] U_1[e] - U_{-1}[] U_1[G[1]]$$

$$x = U_{-1}[] U_1[e] U_{-2}[f] U_2[] U_{-3}[h] U_3[];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$- \in \hbar U_{-1}[H[1]] U_1[] + \epsilon \hbar U_{-1}[] U_1[G[1]]$$

$$x = U_{-1}[h, h, f] U_1[G[2], e, e] U_{-2}[H[-1], f, f] U_2[g] U_{-3}[h, f] U_3[g, g, e];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$2 e^{-5 \epsilon \hbar^2} \in \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] - 4 e^{-4 \epsilon \hbar^2} \in \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] +$$

$$2 e^{-3 \epsilon \hbar^2} \in \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] + 8 e^{-2 \epsilon \hbar^2} \in \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] -$$

$$2 e^{-4 \epsilon \hbar^2} \hbar^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e] + 2 e^{-2 \epsilon \hbar^2} \hbar^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e] -$$

$$2 e^{-9 \epsilon \hbar^2} \in \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] +$$

$$4 e^{-8 \epsilon \hbar^2} \in \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] -$$

$$2 e^{-7 \epsilon \hbar^2} \in \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] -$$

$$8 e^{-6 \epsilon \hbar^2} \in \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] +$$

$$2 e^{-8 \epsilon \hbar^2} \hbar^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] -$$

$$2 e^{-6 \epsilon \hbar^2} \hbar^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] -$$

$$5 e^{-5 \epsilon \hbar^2} \in \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] +$$

$$10 e^{-4 \epsilon \hbar^2} \in \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] -$$

$$5 e^{-3 \epsilon \hbar^2} \in \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] -$$

$$20 e^{-2 \epsilon \hbar^2} \in \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] +$$

$$5 e^{-4 \epsilon \hbar^2} \hbar^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] -$$

$$5 e^{-2 \epsilon \hbar^2} \hbar^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] + \epsilon \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] +$$

$$4 e^{\epsilon \hbar^2} \in \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] +$$

$$3 e^{2 \epsilon \hbar^2} \in \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] -$$

$$\hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] - e^{-\epsilon \hbar^2} \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] +$$

$$e^{\epsilon \hbar^2} \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] +$$

$$e^{2 \epsilon \hbar^2} \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] +$$

$$5 e^{-9 \epsilon \hbar^2} \in \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] -$$

$$10 e^{-8 \epsilon \hbar^2} \in \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] +$$

$$5 e^{-7 \epsilon \hbar^2} \in \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] +$$

$$20 e^{-6 \epsilon \hbar^2} \in \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] -$$

$$5 e^{-8 \epsilon \hbar^2} \hbar^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] +$$

$$5 e^{-6 \epsilon \hbar^2} \hbar^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] -$$

$$3 \in \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - e^{-3 \epsilon \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] -$$

$$5 e^{-2 \epsilon \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - 7 e^{-\epsilon \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] -$$

$$\hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] + e^{-4 \epsilon \hbar^2} \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] +$$

$$2 e^{-3 \epsilon \hbar^2} \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] - 2 e^{-\epsilon \hbar^2} \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] +$$

$$e^{-5 \epsilon \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] +$$

$$\begin{aligned}
 & 4 e^{-4 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] + \\
 & 3 e^{-3 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] - \\
 & e^{-6 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] - \\
 & e^{-5 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & e^{-4 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & e^{-3 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & 4 e^{-5 \in \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & 8 e^{-4 \in \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 4 e^{-3 \in \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 16 e^{-2 \in \hbar^2} \in \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & 4 e^{-4 \in \hbar^2} \hbar U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 4 e^{-2 \in \hbar^2} \hbar U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & \in \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 4 e^{\in \hbar^2} \in \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 3 e^{2 \in \hbar^2} \in \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] + \\
 & U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] + \\
 & e^{-\in \hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & e^{\in \hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & e^{2 \in \hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 4 e^{-9 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 8 e^{-8 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 4 e^{-7 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 16 e^{-6 \in \hbar^2} \in \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 4 e^{-8 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 4 e^{-6 \in \hbar^2} \hbar U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 3 \in \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + e^{-3 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + \\
 & 5 e^{-2 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + 7 e^{-\in \hbar^2} \in \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + \\
 & U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - e^{-4 \in \hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - \\
 & 2 e^{-3 \in \hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] + 2 e^{-\in \hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - \\
 & e^{-5 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & 4 e^{-4 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & 3 e^{-3 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] + \\
 & e^{-6 \in \hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] + \\
 & e^{-5 \in \hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-4 \in \hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-3 \in \hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-5 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & 2 e^{-4 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \\
 & e^{-3 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \\
 & 4 e^{-2 \in \hbar^2} \in \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & e^{-4 \in \hbar^2} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \\
 & e^{-2 \in \hbar^2} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & e^{-9 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \\
 & 2 e^{-8 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \\
 & e^{-7 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \\
 & 4 e^{-6 \in \hbar^2} \in \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \\
 & e^{-8 \in \hbar^2} U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, g, e, e] +
 \end{aligned}$$

The R-Matrix

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