

Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-05/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as e^{Vh}).
2. dm is not meta-associative.
3. R doesn't satisfy YB.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The “degree carrier” is \hbar , and all “coupling constants” are proportional to it.

```
$TD = 3;  $\hbar$  /:  $\hbar^{d-}$  /;  $d > $TD := 0;$ 
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\hbar\eta g}$ and to $H = e^{\hbar\gamma h}$, where $[g, e] = \hbar\gamma e$ and $[h, f] = -\hbar\eta f$.
 Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2, G_1 G_2, e_1 + G_1 e_2, h_1 + h_2, H_1 H_2, f_1 H_2 + f_2)$.
 Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;  
$PBWRule = {G[_] → 1, g → 2, e → 3, H[_] → 4, h → 5, f → 6};
```

```
B[g, e] =  $\hbar\gamma e$ ; B[e, G[n_]] = (e-n $\hbar^2\gamma\eta$  - 1) U[G[n], e]; B[g, G[_]] = 0;  
B[h, f] = - $\hbar\eta f$ ; B[f, H[n_]] = (en $\hbar^2\gamma\eta$  - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on pensieve://Projects/UEA/.

```
B[0, _] = 0; B[_ , 0] = 0;  
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];  
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];  
B[x_Plus, y_] := B[# , y] & /@ x;  
B[x_, y_Plus] := B[x, #] & /@ y;  
B[x_, x_] = 0;  
B[y_, x_] := Expand[-B[x, y]];
```

```

x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^p_] := U_i[x[n p]];
UU_i_[x_^p_] := UU_i@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => U_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U_[] ..], Expand];
USimp[ε_] := Expand[ε];

```

```

m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_i->j_[ε_] := ε /. U_i -> U_j;

```

```

m_i->j->k_[c_. U_i[x_] U_j[]] := c U_k[x];
m_i->j->k_[c_. U_i[] U_j[y_]] := c U_k[y];
m_i->j->k_[c_. U_i[xx_, x_[n1_]] U_j[x_[n2_], yy_]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j->k];
m_i->j->k_[c_. U_i[xx_, x_] U_j[y_, yy_]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];

```

```

Supp[ε_] := Union@Cases[{ε}, U_i[___] -> i, ∞];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
  {Sx = Supp[x], Sy = Supp[y], is, σ, z},
  If[MatchQ[Sx ∪ Sy, {_Integer ...}] && Min[Sx ∪ Sy] < 0,
    is = Abs[Sx] ∩ Abs[Sy];
    z = x; Do[z = m_i->σ@i[m_i->σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_σ@i,i->i[z], {i, is}];
    z,
    (* else *) is = Sx ∩ Sy;
    z = x; Do[z = m_i->σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = m_σ@i,i->i[z], {i, is}];
    z
  ]
];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[func_] := Normal@Series[func, {h, 0, $TD}];
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U[u___] => Cases[{u}, r_Rule], ∞];
  vars = First/@rules; elems = Last/@rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ -> c_) => c (
    specs /. MapThread[{(u1 -> _) => u2} &, {vars, ps, elems}] /. U_i_ => UU_i
  )]
]

```

The 2D Lie BiAlgebra Pencil, Testing

$O[U_1[x \rightarrow g], e^{\hbar \eta x}]$

$$U_1[] + \eta \hbar U_1[g] + \frac{1}{2} \eta^2 \hbar^2 U_1[g, g] + \frac{1}{6} \eta^3 \hbar^3 U_1[g, g, g]$$

$USimp@With[{G = O[U_1[x \rightarrow g], e^{\hbar \eta x}]}, UB[U_1[e], G] - O[e^{-\hbar^2 \gamma \eta} - 1] G ** U_1[e]]$

0

$B[e, G[3]]$

$$(-1 + e^{-3 \gamma \eta \hbar^2}) U[G[3], e]$$

$\$TD = 6;$

$USimp@With[{H = O[U_1[x \rightarrow h], e^{\hbar \gamma x}]}, UB[U_1[f], H] - O[e^{\hbar^2 \gamma \eta} - 1] H ** U_1[f]]$

0

$x = U_1[g, G[2], e, e, e] U_2[g, g, e] U_3[g, g, G[-3], e];$

$$(x // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) - (x // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})$$

0

$x = U_1[h, H[2], f, f, f] U_2[h, h, f] U_3[h, h, H[-3], f];$

$$(x // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) - (x // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})$$

0

The Co-Product and Co-Associativity

```

qΔi→j,k[ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[g, xS____] =>
    (USimp[(Uj[g] Uk[] + Uj[] Uk[g]) qΔi→tj,tk[Ui[xS]]] // mj,tj→j // mk,tk→k),
  Ui[G[n-], xS____] => (USimp[Uj[G[n]] Uk[G[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[e, xS____] => (USimp[(Uj[e] Uk[G[1]] + Uj[] Uk[e]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[h, xS____] => (USimp[(Uj[h] Uk[] + Uj[] Uk[h]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[H[n-], xS____] => (USimp[Uj[H[n]] Uk[H[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[f, xS____] => (USimp[(Uj[f] Uk[] + Uj[H[1]] Uk[f]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k)
}];

```

```

qΔi→j,k,l[ε-] := ε // qΔi→j,k // qΔk→l

```

```

U1[e] // qΔ1→1,2
U1[] U2[e] + U1[e] U2[G[1]]

{lhs = U1[e] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[e] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
{U1[] U2[] U3[e] + U1[] U2[e] U3[G[1]] + U1[e] U2[G[1]] U3[G[1]],
 U1[] U2[] U3[e] + U1[] U2[e] U3[G[1]] + U1[e] U2[G[1]] U3[G[1]], True}

U1[f] // qΔ1→1,2
U1[f] U2[] + U1[H[1]] U2[f]

{lhs = U1[f] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[f] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
{U1[f] U2[] U3[] + U1[H[1]] U2[f] U3[] + U1[H[1]] U2[H[1]] U3[f],
 U1[f] U2[] U3[] + U1[H[1]] U2[f] U3[] + U1[H[1]] U2[H[1]] U3[f], True}

x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // qΔ1→1,2) - (x // qΔ2→3,4 // qΔ1→1,2 // m1,3→1 // m2,4→2)
0

x = U1[h, H[2], f, f, f] U2[h, h, H[-3], f];
(x // m1,2→1 // qΔ1→1,2) - (x // qΔ2→3,4 // qΔ1→1,2 // m1,3→1 // m2,4→2)
0

```

The Antipode

Why o why this annoyance of left-vs-right?

```
S[g] = -g; S[G[n_]] := G[-n]; S[e] = -eħ2γη U[G[-1], e];
S[h] = -h; S[H[n_]] := H[-n]; S[f] = -U[H[-1], f];
Si[ε_] := Module[{ti}, USimp[
  ε /. Ui[X_, xS_____] => mti,i->i[Expand[UUi[S[X]] Sti[Uti[xS]]]]
];
```

{lhs = S₁[U₁[e]], rhs = -U₁[e] ** U₁[G[-1]], lhs == rhs}

{-e^{γ η ħ²} U₁[G[-1], e], -e^{γ η ħ²} U₁[G[-1], e], True}

U₁[e] // S₁ // S₁

e^{γ η ħ²} U₁[e]

U₁[f] // S₁ // S₁

e^{γ η ħ²} U₁[f]

S₁[U₁[g, G[3], e, e]]

2 e^{9 γ η ħ²} γ ħ U₁[G[-5], e, e] - e^{9 γ η ħ²} U₁[G[-5], g, e, e]

U₁[g, G[3], e, e] // qΔ_{1->1,2}

U₁[G[3], g, e, e] U₂[G[5]] + U₁[G[3], g, e] U₂[G[4], e] + e^{-γ η ħ²} U₁[G[3], g, e] U₂[G[4], e] +
 U₁[G[3], e, e] U₂[G[5], g] + U₁[G[3], g] U₂[G[3], e, e] + U₁[G[3], e] U₂[G[4], g, e] +
 e^{-γ η ħ²} U₁[G[3], e] U₂[G[4], g, e] + U₁[G[3]] U₂[G[3], g, e, e]

U₁[g, G[3], e, e] // qΔ_{1->1,2} // S₂

U₁[G[3], g, e, e] U₂[G[-5]] -
 e^{4 γ η ħ²} γ ħ U₁[G[3], e] U₂[G[-5], e] - e^{5 γ η ħ²} γ ħ U₁[G[3], e] U₂[G[-5], e] -
 e^{4 γ η ħ²} U₁[G[3], g, e] U₂[G[-5], e] - e^{5 γ η ħ²} U₁[G[3], g, e] U₂[G[-5], e] -
 U₁[G[3], e, e] U₂[G[-5], g] + 2 e^{9 γ η ħ²} γ ħ U₁[G[3]] U₂[G[-5], e, e] +
 e^{9 γ η ħ²} U₁[G[3], g] U₂[G[-5], e, e] + e^{4 γ η ħ²} U₁[G[3], e] U₂[G[-5], g, e] +
 e^{5 γ η ħ²} U₁[G[3], e] U₂[G[-5], g, e] - e^{9 γ η ħ²} U₁[G[3]] U₂[G[-5], g, e, e]

test = U₁[g, G[3], e, e];

{test // qΔ_{1->1,2} // S₂ // m_{1,2->1}, test // qΔ_{1->1,2} // S₂ // m_{2,1->1},
 test // qΔ_{1->1,2} // S₁ // m_{1,2->1}, test // qΔ_{1->1,2} // S₁ // m_{2,1->1}}

{0, 0, 0, 2 e^{6 γ η ħ²} γ ħ U₁[e, e] - 2 e^{7 γ η ħ²} γ ħ U₁[e, e] - 2 e^{8 γ η ħ²} γ ħ U₁[e, e] + 2 e^{9 γ η ħ²} γ ħ U₁[e, e]}

test = U₁[h, H[3], f, f];

{test // qΔ_{1->1,2} // S₂ // m_{1,2->1}, test // qΔ_{1->1,2} // S₂ // m_{2,1->1},
 test // qΔ_{1->1,2} // S₁ // m_{1,2->1}, test // qΔ_{1->1,2} // S₁ // m_{2,1->1}}

{0, 0, 0, -2 e^{-10 γ η ħ²} η ħ U₁[H[-2], f, f] + 2 e^{-9 γ η ħ²} η ħ U₁[H[-2], f, f] +
 2 e^{-8 γ η ħ²} η ħ U₁[H[-2], f, f] - 2 e^{-7 γ η ħ²} η ħ U₁[H[-2], f, f]}

x = U₁[h, H[2], f, f, f] U₂[h, h, H[-3], f];

(x // m_{1,2->1} // S₁) - (x // S₁ // S₂ // m_{2,1->1})

0

```
x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
0
```

```
x = U1[];
(x // qΔ1→1,2 // S1 // m1,2→1)
U1[]
```

```
x = U1[];
(x // qΔ1→1,2 // S2 // m1,2→1)
U1[]
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S1 // m1,2→1)
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S2 // m1,2→1)
0
```

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[]] = 1;
P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1;
P[U[], U[[_]]] = P[U[[_]], U[]] = 0;
(
  P[U[g], U[h]] = 1          P[U[g], U[H[n_]]] = n ħ γ          P[U[g], U[f]] = 0
  P[U[G[n_]], U[h]] = n ħ η P[U[G[n_]], U[H[m_]]] = en m ħ2 η γ P[U[G[_]], U[f]] = 0
  P[U[e], U[h]] = 0          P[U[e], U[H[_]]] = 0          P[U[e], U[f]] = 1
);
```

```
Pi,j[ε-] := USimp[ε / . Ui[xs____] Uj[ys____] → P[U[xs], U[ys]]];
```

```
t = Ui[g] Uj[e] Uk[f];
{mi,j→i[t] - mj,i→i[t], qΔk→k,1[t] - qΔk→1,k[t]}
{γ ħ Ui[e] Uk[f], Ui[g] Uj[e] Uk[f] U1[] -
  Ui[g] Uj[e] Uk[] U1[f] + Ui[g] Uj[e] Uk[H[1]] U1[f] - Ui[g] Uj[e] Uk[f] U1[H[1]]}
```

```

t = Ui[g] Uj[e] Uk[f];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1}
{γ ħ, γ ħ}

Table[t = Ui[xi] Uj[xj] Uk[yk];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1},
{xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {γ ħ, γ ħ}}, {{{0, 0}, {-γ ħ, -γ ħ}}, {{0, 0}, {0, 0}}}}

Table[t = Ui[xi] Uk[yk] Ul[yl];
{(qΔi→i,j[t] - qΔi→j,i[t]) // Pi,k // Pj,1, (mk,1→k[t] - m1,k→k[t]) // Pi,k},
{xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{{0, 0}, {-η ħ, -η ħ}}, {{η ħ, η ħ}, {0, 0}}}}

```

General Pairings

The pairing sequence: (one,one) (above), (many,one), (many,many).

```

P[U[x_, xs_], U[y_]] := P[U[x, xs], U[y]] =
Module[{i, j, k, l}, USimp[Ui[x] Uj[xs] qΔk→k,1[Uk[y]]] // Pi,k // Pj,1];
P[U[xs_], U[y_, ys_]] := P[U[xs], U[y, ys]] =
Module[{i, j, k, l}, USimp[qΔi→i,j[Ui[xs]] Uk[y] Ul[ys]] // Pi,k // Pj,1];

```

```

{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
{0, γ ħ, 0}

```

```
P[U[e], U[f, f]]
```

```
0
```

```
P[U[e, e], U[f, f]]
```

```
1 + eγ η ħ2
```

```
lhs = Factor@Table[P[U@@Table[e, {n}], U@@Table[f, {n}]], {n, 7}]
```

```

{1, 1 + eγ η ħ2, (1 + eγ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2), (1 + eγ η ħ2)2 (1 + e2 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2),
(1 + eγ η ħ2)2 (1 + e2 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2 + e3 γ η ħ2 + e4 γ η ħ2), (1 + eγ η ħ2)3
(1 + e2 γ η ħ2) (1 - eγ η ħ2 + e2 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2)2 (1 + eγ η ħ2 + e2 γ η ħ2 + e3 γ η ħ2 + e4 γ η ħ2),
(1 + eγ η ħ2)3 (1 + e2 γ η ħ2) (1 - eγ η ħ2 + e2 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2)2
(1 + eγ η ħ2 + e2 γ η ħ2 + e3 γ η ħ2 + e4 γ η ħ2) (1 + eγ η ħ2 + e2 γ η ħ2 + e3 γ η ħ2 + e4 γ η ħ2 + e5 γ η ħ2 + e6 γ η ħ2)}

```

rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^{ħ²γη}], {n, 7}]

$$\left\{ 1, 1 + e^{\gamma \eta \hbar^2}, \left(1 + e^{\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right), \left(1 + e^{\gamma \eta \hbar^2}\right)^2 \left(1 + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right), \right. \\ \left. \left(1 + e^{\gamma \eta \hbar^2}\right)^2 \left(1 + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2} + e^{3\gamma \eta \hbar^2} + e^{4\gamma \eta \hbar^2}\right), \left(1 + e^{\gamma \eta \hbar^2}\right)^3 \right. \\ \left. \left(1 + e^{2\gamma \eta \hbar^2}\right) \left(1 - e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right)^2 \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2} + e^{3\gamma \eta \hbar^2} + e^{4\gamma \eta \hbar^2}\right), \right. \\ \left. \left(1 + e^{\gamma \eta \hbar^2}\right)^3 \left(1 + e^{2\gamma \eta \hbar^2}\right) \left(1 - e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right)^2 \right. \\ \left. \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2} + e^{3\gamma \eta \hbar^2} + e^{4\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2} + e^{3\gamma \eta \hbar^2} + e^{4\gamma \eta \hbar^2} + e^{5\gamma \eta \hbar^2} + e^{6\gamma \eta \hbar^2}\right) \right\}$$

lhs == rhs

True

P[U[g, g, g, g, g], U[h, h, h, h, h]]

120

P[U[g, g, g, g, g, e, e, e, e], U[h, h, h, h, h, f, f, f, f]] // Factor

$$120 \left(1 + e^{\gamma \eta \hbar^2}\right)^2 \left(1 + e^{2\gamma \eta \hbar^2}\right) \left(1 + e^{\gamma \eta \hbar^2} + e^{2\gamma \eta \hbar^2}\right)$$

x = U₁[g, G[-3], e, e] U₂[g, g, G[1], e] U₃[h, H[2], f, f, f];

(x // m_{1,2→1} // P_{1,3}) - (x // qΔ_{3→3,4} // P_{1,3} // P_{2,4})

0

x = U₁[h, H[-3], f, f] U₂[h, h, H[1], f] U₃[g, G[2], e, e, e];

(x // m_{1,2→1} // P_{3,1}) - (x // qΔ_{3→3,4} // P_{3,1} // P_{4,2})

0

x = U₁[g, g, G[-3], e, e, e] U₂[h, h, H[2], f, f, f];

(x // S₁ // P_{1,2}) - (x // S₂ // P_{1,2})

0

The Double

```
dmi,j→k[E-] := Module[{t1, t2, t3, h1, h2, h3},
  E // qΔi→h1,h2,h3 // Sh1 // qΔ-j→t1,t2,t3 // Ph1,t1 // Ph3,t3 // mh2,j→k // m-i,t2→-k]
```

U₋₁[[] U₁[g] U₋₂[h] U₂[[] // dm_{1,2→1}

U₋₁[h] U₁[g]

U₋₁[[] U₁[g] U₋₂[f] U₂[[] // dm_{1,2→1}

-γ ħ U₋₁[f] U₁[[] + U₋₁[f] U₁[g]

U₋₁[[] U₁[G[1]] U₋₂[f] U₂[[] // dm_{1,2→1}

e^{-γ η ħ²} U₋₁[f] U₁[G[1]]

$$U_{-1}[] U_1[e] U_{-2}[h] U_2[] // dm_{1,2 \rightarrow 1}$$

$$\eta \hbar U_{-1}[] U_1[e] + U_{-1}[h] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[H[1]] U_2[] // dm_{1,2 \rightarrow 1}$$

$$e^{\gamma \eta \hbar^2} U_{-1}[H[1]] U_1[e]$$

$$U_{-1}[] U_1[e] U_{-2}[f] U_2[] // dm_{1,2 \rightarrow 1}$$

$$U_{-1}[H[1]] U_1[] + U_{-1}[f] U_1[e] - U_{-1}[] U_1[G[1]]$$

$$x = U_{-1}[] U_1[e] U_{-2}[f] U_2[] U_{-3}[h] U_3[];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$-\eta \hbar U_{-1}[H[1]] U_1[] + \eta \hbar U_{-1}[] U_1[G[1]]$$

$$x = U_{-1}[h, h, f] U_1[G[2], e, e] U_{-2}[H[-1], f, f] U_2[g] U_{-3}[h, f] U_3[g, g, e];$$

$$(x // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (x // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1})$$

$$\begin{aligned} & 2 e^{-5 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] - \\ & 4 e^{-4 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] + \\ & 2 e^{-3 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] + \\ & 8 e^{-2 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[h, h, f, f, f] U_1[G[2], e, e] - \\ & 2 e^{-4 \gamma \eta \hbar^2} \gamma^3 \hbar^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e] + \\ & 2 e^{-2 \gamma \eta \hbar^2} \gamma^3 \hbar^3 U_{-1}[h, h, h, f, f, f] U_1[G[2], e, e] - \\ & 2 e^{-9 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] + \\ & 4 e^{-8 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] - \\ & 2 e^{-7 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] - \\ & 8 e^{-6 \gamma \eta \hbar^2} \gamma^3 \eta \hbar^4 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], e, e] + \\ & 2 e^{-8 \gamma \eta \hbar^2} \gamma^3 \hbar^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] - \\ & 2 e^{-6 \gamma \eta \hbar^2} \gamma^3 \hbar^3 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], e, e] - \\ & 5 e^{-5 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] + \\ & 10 e^{-4 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] - \\ & 5 e^{-3 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] - \\ & 20 e^{-2 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[h, h, f, f, f] U_1[G[2], g, e, e] + \\ & 5 e^{-4 \gamma \eta \hbar^2} \gamma^2 \hbar^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] - \\ & 5 e^{-2 \gamma \eta \hbar^2} \gamma^2 \hbar^2 U_{-1}[h, h, h, f, f, f] U_1[G[2], g, e, e] + \\ & \gamma \eta \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] + \\ & 4 e^{\gamma \eta \hbar^2} \gamma \eta \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] + \\ & 3 e^{2 \gamma \eta \hbar^2} \gamma \eta \hbar^2 U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, e] - \\ & \gamma \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] - \\ & e^{-\gamma \eta \hbar^2} \gamma \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \\ & e^{\gamma \eta \hbar^2} \gamma \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \\ & e^{2 \gamma \eta \hbar^2} \gamma \hbar U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, e] + \\ & 5 e^{-9 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] - \\ & 10 e^{-8 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] + \\ & 5 e^{-7 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] + \\ & 20 e^{-6 \gamma \eta \hbar^2} \gamma^2 \eta \hbar^3 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, e, e] - \\ & 5 e^{-8 \gamma \eta \hbar^2} \gamma^2 \hbar^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] + \\ & 5 e^{-6 \gamma \eta \hbar^2} \gamma^2 \hbar^2 U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, e, e] - \end{aligned}$$

$$\begin{aligned}
 & 3 \gamma \eta \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - e^{-3\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - \\
 & 5 e^{-2\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - \\
 & 7 e^{-\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f] U_1[G[3], g, g, e] - \\
 & \gamma \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] + e^{-4\gamma\eta\hbar^2} \gamma \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] + \\
 & 2 e^{-3\gamma\eta\hbar^2} \gamma \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] - \\
 & 2 e^{-\gamma\eta\hbar^2} \gamma \hbar U_{-1}[h, h, h, f, f] U_1[G[3], g, g, e] + \\
 & e^{-5\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] + \\
 & 4 e^{-4\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] + \\
 & 3 e^{-3\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, e] - \\
 & e^{-6\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] - \\
 & e^{-5\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & e^{-4\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & e^{-3\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, e] + \\
 & 4 e^{-5\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & 8 e^{-4\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 4 e^{-3\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 16 e^{-2\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & 4 e^{-4\gamma\eta\hbar^2} \gamma \hbar U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] + \\
 & 4 e^{-2\gamma\eta\hbar^2} \gamma \hbar U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, e, e] - \\
 & \eta \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 4 e^{\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 3 e^{2\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[1], h, h, f, f] U_1[G[2], g, g, g, e] + \\
 & U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] + \\
 & e^{-\gamma\eta\hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & e^{\gamma\eta\hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & e^{2\gamma\eta\hbar^2} U_{-1}[H[1], h, h, h, f, f] U_1[G[2], g, g, g, e] - \\
 & 4 e^{-9\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 8 e^{-8\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 4 e^{-7\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 16 e^{-6\gamma\eta\hbar^2} \gamma \eta \hbar^2 U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 4 e^{-8\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] - \\
 & 4 e^{-6\gamma\eta\hbar^2} \gamma \hbar U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, e, e] + \\
 & 3 \eta \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + e^{-3\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + \\
 & 5 e^{-2\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + \\
 & 7 e^{-\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f] U_1[G[3], g, g, g, e] + U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - \\
 & e^{-4\gamma\eta\hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - 2 e^{-3\gamma\eta\hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] + \\
 & 2 e^{-\gamma\eta\hbar^2} U_{-1}[h, h, h, f, f] U_1[G[3], g, g, g, e] - \\
 & e^{-5\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & 4 e^{-4\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & 3 e^{-3\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f] U_1[G[4], g, g, g, e] + \\
 & e^{-6\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] + \\
 & e^{-5\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-4\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-3\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f] U_1[G[4], g, g, g, e] - \\
 & e^{-5\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & 2 e^{-4\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \\
 & e^{-3\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] -
 \end{aligned}$$

$$\begin{aligned}
 & 4 e^{-2\gamma\eta\hbar^2} \eta \hbar U_{-1}[h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & e^{-4\gamma\eta\hbar^2} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] - \\
 & e^{-2\gamma\eta\hbar^2} U_{-1}[h, h, h, f, f, f] U_1[G[2], g, g, g, e, e] + \\
 & e^{-9\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \\
 & 2 e^{-8\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \\
 & e^{-7\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \\
 & 4 e^{-6\gamma\eta\hbar^2} \eta \hbar U_{-1}[H[-1], h, h, f, f, f] U_1[G[3], g, g, g, e, e] - \\
 & e^{-8\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, g, e, e] + \\
 & e^{-6\gamma\eta\hbar^2} U_{-1}[H[-1], h, h, h, f, f, f] U_1[G[3], g, g, g, e, e]
 \end{aligned}$$

The R-Matrix

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