

Pensieve header: Figuring out the double of the 2D pencil; older versions under "MetaDouble-\*\*".

Issues:

1. Improve DUForm.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The "degree carrier" is  $\hbar$ , and all "coupling constants" are proportional to it.

```
$TD = 3;  $\hbar$  /:  $\hbar^{d.}$  /;  $d > $TD := 0;$ 
```

### The 2D Lie BiAlgebra Pencil

We hope to stick to  $A = e^{\hbar\beta a}$  and to  $B = e^{\hbar\alpha b}$ , where  $[a, x] = \alpha x$  and  $[b, y] = -\beta y$ .

Also,  $\Delta_{12}(a, A, x, b, B, y) = (a_1 + a_2, A_1 A_2, x_1 + A_1 x_2, b_1 + b_2, B_1 B_2, y_1 B_2 + y_2)$ .

Also,  $(a, x)$  and  $\hbar(b, y)$  are dual bases.

```
AlgebraAtom = a | A[_] | x | b | B[_] | y;
$PBWRule = {A[_] -> 1, a -> 2, x -> 3, y -> 4, B[_] -> 5, b -> 6};
```

```
B[a, x] =  $\alpha x$ ; B[x, A[n_]] = (e $\hbar\alpha\beta$  - 1) U[A[n], x]; B[a, A[_]] = 0;
B[y, b] =  $\beta y$ ; B[B[n_], y] = (e $\hbar\alpha\beta$  - 1) U[y, B[n]]; B[b, B[_]] = 0;
 $\Delta[a] = U_1[a] U_2[] + U_1[] U_2[a]$ ;  $\Delta[A[n_]] := U_1[A[n]] U_2[A[n]]$ ;
 $\Delta[x] = U_1[x] U_2[] + U_1[A[1]] U_2[x]$ ;
 $\Delta[b] = U_1[b] U_2[] + U_1[] U_2[b]$ ;  $\Delta[B[n_]] := U_1[B[n]] U_2[B[n]]$ ;
 $\Delta[y] = U_1[y] U_2[B[1]] + U_1[] U_2[y]$ ;
S[a] = -a; S[A[n_]] := A[-n]; S[x] = -U[A[-1], x];
S[b] = -b; S[B[n_]] := B[-n]; S[y] = -U[y, B[-1]];
Si[a] = -a; Si[A[n_]] := A[-n]; Si[x] = -U[x, A[-1]];
Si[b] = -b; Si[B[n_]] := B[-n]; Si[y] = -U[B[-1], y];
(* This extra line is annoying *)
```

```
ExpandAB[ $\mathcal{E}$ ] := Expand@Normal@0[ $\mathcal{E}$  // . {
  c_. U_i[_lambda_, A[n_],  $\rho$ _] =>
  Expand[c Sum[ $\frac{(-1)^d \hbar^d \beta^d n^d}{d!}$  U_i[ $\lambda$ , Sequence@@Table[a, {d}],  $\rho$ ], {d, 0, $TD}]],
  c_. U_i[_lambda_, B[n_],  $\rho$ _] => Expand[
  c Sum[ $\frac{(-1)^d \hbar^d \alpha^d n^d}{d!}$  U_i[ $\lambda$ , Sequence@@Table[b, {d}],  $\rho$ ], {d, 0, $TD}]]
}]
```

### UEA with provisional modification

This section is based on pensieve://Projects/UEA/.

```

B[0, _] = 0; B[_, 0] = 0;
B[c_*x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_*x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];

```

```

x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^-] := U_i[x[n p]];
UU_i_[x^-] := UU_i_@@Table[x, {p}];
UU_i_[ε_] := ε /. {
  U[xs_] => UU_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i_[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε_] := Expand[ε];

```

```

m_s_[0] = 0;
m_s_[x_Plus] := m_s_/@x;
m_i->j_[ε_] := ε /. U_i -> U_j;

```

```

m_i,j->k_[c_. U_i[x___] U_j[]] := c U_k[x];
m_i,j->k_[c_. U_i[] U_j[y___]] := c U_k[y];
m_i,j->k_[c_. U_i[xx___, x_[n1_]] U_j[x_[n2_], yy___]] :=
  USimp[c If[TrueQ[n1 + n2 == 0], U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j->k];
m_i,j->k_[c_. U_i[xx___, x_] U_j[y_, yy___]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];

```

```

Supp[ε_] := Union@Cases[{ε}, U_i[___] => i, ∞];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
  {Sx = Supp[x], Sy = Supp[y], is, σ, z},
  If[MatchQ[Sx ∪ Sy, {_Integer ...}] && Min[Sx ∪ Sy] < 0,
    is = Abs[Sx] ∩ Abs[Sy];
    z = x; Do[z = m-i→-σ@i[mi→σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dmσ@i, i→i[z], {i, is}];
    z,
    (* else *) is = Sx ∩ Sy;
    z = x; Do[z = mi→σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = mσ@i, i→i[z], {i, is}];
    z
  ]
];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[func_] := Normal@(func + O[ħ] $TD+1);
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U[u___] => Cases[{u}, r_Rule], ∞];
  vars = First/@rules; elems = Last/@rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ -> c_) => c (
    specs /. MapThread[{(#1 -> _) => #3^#2} &, {vars, ps, elems}] /. Ui -> UUi
  )]
]

```

## The 2D Lie BiAlgebra Pencil, Testing

$$O[U_1[a \to a], e^{\hbar \beta a}]$$

$$U_1[] + \beta \hbar U_1[a] + \frac{1}{2} \beta^2 \hbar^2 U_1[a, a] + \frac{1}{6} \beta^3 \hbar^3 U_1[a, a, a]$$

$$USimp@With[{An = O[U_1[a \to a], e^{-\hbar \beta a}]}, UB[U_1[x], An] - O[e^{\hbar \alpha \beta} - 1] An ** U_1[x]}$$

0

$$B[x, A[3]]$$

$$(-1 + e^{3\alpha\beta\hbar}) U[A[3], x]$$

$$\$TD = 6;$$

$$USimp@With[{Bn = O[U_1[b \to b], e^{-\hbar \alpha b}]}, UB[Bn, U_1[y]] - O[e^{\hbar \alpha \beta} - 1] U_1[y] ** Bn}$$

0

$$z = U_1[a, A[2], x, x, x] U_2[a, a, x] U_3[a, a, A[-3], x];$$

$$(z // m_{1,2 \to 1} // m_{1,3 \to 1}) - (z // m_{2,3 \to 2} // m_{1,2 \to 1})$$

0

```

z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];
(z // m1,2→1 // m1,3→1) - (z // m2,3→2 // m1,2→1)
0

```

## The Co-Product and Co-Associativity

```

Δi→j-,k- [ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[f-, fs-] :=
    (USimp[Δ[f] /. {U1 → Uj, U2 → Uk}) Δi→tj,tk[Ui[fs]] // m_j,tj→j // m_k,tk→k
}];
Δi→j-,k-,l- [ε-] := ε // Δi→j,k // Δk→l

```

```

Δ1→1,2[U1[#]] & /@ {a, A[7], x, y, b, B[-3]}

```

```

{U1[a] U2[] + U1[] U2[a], U1[A[7]] U2[A[7]], U1[x] U2[] + U1[A[1]] U2[x],
 U1[] U2[y] + U1[y] U2[B[1]], U1[b] U2[] + U1[] U2[b], U1[B[-3]] U2[B[-3]]}

```

```

{lhs = U1[x] // Δ1→1,2 // Δ2→2,3, rhs = U1[x] // Δ1→1,3 // Δ1→1,2, lhs == rhs}

```

```

{U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x],
 U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x], True}

```

```

U1[y] // Δ1→1,2

```

```

U1[] U2[y] + U1[y] U2[B[1]]

```

```

{lhs = U1[y] // Δ1→1,2 // Δ2→2,3, rhs = U1[y] // Δ1→1,3 // Δ1→1,2, lhs == rhs}

```

```

{U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]],
 U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]], True}

```

```

z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x];

```

```

(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)

```

```

0

```

```

z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];

```

```

(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)

```

```

0

```

## The Antipode

```

Si- [ε-] := Module[{ti}, USimp[
  ε /. Ui[x-, xs-] := mti,i-i[Expand[UUi[S[x]] S- ti[Uti[xs]]]]
]];
Si_ [ε-] := Module[{ti}, USimp[
  ε /. Ui[x-, xs-] := mti,i-i[Expand[UUi[Si[x]] Si- ti[Uti[xs]]]]
]];

```

```

{U1[x] // S1 // Si, U1[y] // S1 // Si}
{e^{\alpha\beta\hbar} U1[x], e^{\alpha\beta\hbar} U1[y]}

{U1[x] // S1 // Si1, U1[y] // S1 // Si1}
{U1[x], U1[y]}

{z = U1[]; (z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}), z = U1[]; (z // \Delta_{1\to 2} // S2 // m_{1,2\to 1})}
{U1[], U1[]}

z = U1[a, A[3], x, x];
{z // \Delta_{1\to 2} // S2 // m_{1,2\to 1}, z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}}
{0, 0}

z = U1[y, y, y, b, B[-3]];
{z // \Delta_{1\to 2} // S2 // m_{1,2\to 1}, z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}}
{0, 0}

{z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x]; (z // m_{1,2\to 1} // S1) - (z // S1 // S2 // m_{2,1\to 1}),
z = U1[y, y, y, b, B[2]] U2[y, b, b, b, B[6]]; (z // m_{1,2\to 1} // S1) - (z // S1 // S2 // m_{2,1\to 1})}
{0, 0}

{z = U1[a, A[2], x, x, x]; (z // S1 // \Delta_{1\to 2,1}) - (z // \Delta_{1\to 2} // S1 // S2),
z = U1[y, y, y, b, B[2]]; (z // S1 // \Delta_{1\to 2,1}) - (z // \Delta_{1\to 2} // S1 // S2)}
}
{0, 0}

```

## The Pairing

$$\begin{aligned}
 &P[U[], U[B[_]]] = P[U[A[_]], U[]] = 1; \\
 &P[U[], U[[_]]] = P[U[[_]], U[]] = 0; \\
 &\left( \begin{array}{lll}
 P[U[a], U[b]] = \hbar^{-1} & P[U[a], U[B[n_]]] = -n\alpha & P[U[a], U[y]] = 0 \\
 P[U[A[n_]], U[b]] = -n\beta & P[U[A[n_]], U[B[m_]]] = e^{nm\hbar\alpha\beta} & P[U[A[_]], U[y]] = 0 \\
 P[U[x], U[b]] = 0 & P[U[x], U[B[_]]] = 0 & P[U[x], U[y]] = \hbar^{-1}
 \end{array} \right);
 \end{aligned}$$

$$\begin{aligned}
 &P[U[], U[]] = 1; \\
 &P_{i,j}[\mathcal{E}] := \text{USimp}[\mathcal{E} /. U_i[xs\_]\ U_j[ys\_]\ \to\ P[U[xs], U[ys]]];
 \end{aligned}$$

The pairing sequence: (one,one) (above), (many,one), (many,many).

$$\begin{aligned}
 &P[U[x\_], xs\_], U[y\_]] := P[U[x, xs], U[y]] = \\
 &\quad \text{Module}[\{i, j, k, l\}, \text{USimp}[U_i[x] \ U_j[xs] \ \Delta_{k\to l}[U_k[y]]] // P_{i,k} // P_{j,l}]; \\
 &P[U[xs\_], U[y\_], ys\_]] := P[U[xs], U[y, ys]] = \\
 &\quad \text{Module}[\{i, j, k, l\}, \text{USimp}[\Delta_{i\to j}[U_i[xs]] \ U_k[y] \ U_l[ys]] // P_{i,k} // P_{j,l}];
 \end{aligned}$$

```
z = Ui[a] Uj[x] Uk[y];
{mi,j→i[z] - mj,i→i[z], Δk→k,1[z] - Δk→1,k[z]}
{α Ui[x] Uk[y], -Ui[a] Uj[x] Uk[y] U1[ ] +
  Ui[a] Uj[x] Uk[ ] U1[y] - Ui[a] Uj[x] Uk[B[1]] U1[y] + Ui[a] Uj[x] Uk[y] U1[B[1]]}
```

```
Table[z = Ui[xi] Uj[xj] Uk[yk];
  {(mi,j→i[z] - mj,i→i[z]) // Pi,k, (Δk→k,1[z] - Δk→1,k[z]) // Pi,k // Pj,1},
  {xi, {a, x}}, {xj, {a, x}}, {yk, {b, y}}]
{{{({0, 0}), {0, 0}}, {{0, 0}, {α/h, α/h}}}, {{{({0, 0}), {-α/h, -α/h}}, {{0, 0}, {0, 0}}}}
```

```
Table[z = Ui[xi] Uk[yk] U1[y1];
  {(Δi→i,j[z] - Δi→j,i[z]) // Pi,k // Pj,1, (mk,1→k[z] - m1,k→k[z]) // Pi,k},
  {xi, {a, x}}, {yk, {b, y}}, {y1, {b, y}}]
{{{({0, 0}), {0, 0}}, {{0, 0}, {0, 0}}}, {{{({0, 0}), {-β/h, -β/h}}, {{{β/h, β/h}, {0, 0}}}}
```

```
lhs = Factor@Table[hn P[U@@Table[x, {n}], U@@Table[y, {n}]], {n, $TD = 7}]
{1, 1 + eαβh, (1 + eαβh) (1 + eαβh + e2αβh), (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh),
  (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh),
  (1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2 (1 + eαβh + e2αβh + e3αβh + e4αβh),
  (1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2
  (1 + eαβh + e2αβh + e3αβh + e4αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh + e5αβh + e6αβh)}
```

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, ehαβ], {n, $TD = 7}]
{1, 1 + eαβh, (1 + eαβh) (1 + eαβh + e2αβh), (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh),
  (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh),
  (1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2 (1 + eαβh + e2αβh + e3αβh + e4αβh),
  (1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2
  (1 + eαβh + e2αβh + e3αβh + e4αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh + e5αβh + e6αβh)}
```

```
MapThread[Equal, {lhs, rhs}]
{True, True, True, True, True, True, True}
```

```
P[U[a, a, a, a, a], U[b, b, b, b, b]]
```

$$\frac{120}{h^5}$$

```
Table[P[z1, z2],
  {z1, {U[], U[a], U[x], U[a, a], U[a, x], U[x, x],
    U[a, a, a], U[a, a, x], U[a, x, x], U[x, x, x]}}, {z2, {U[], U[b], U[y],
    U[b, b], U[y, b], U[y, y], U[b, b, b], U[y, b, b], U[y, y, b], U[y, y, y]}
] //
MatrixForm
(
  1 0 0 0 0 0 0 0 0 0
  0 1/h 0 0 0 0 0 0 0 0
  0 0 1/h 0 0 0 0 0 0 0
  0 0 0 2/h^2 0 0 0 0 0 0
  0 0 0 0 1/h^2 0 0 0 0 0
  0 0 0 0 0 1/h^2 + e^{\alpha \beta h}/h^2 0 0 0 0
  0 0 0 0 0 0 6/h^3 0 0 0
  0 0 0 0 0 0 0 2/h^3 0 0
  0 0 0 0 0 0 0 0 1/h^3 + e^{\alpha \beta h}/h^3 0
  0 0 0 0 0 0 0 0 0 1/h^3 + 2e^{\alpha \beta h}/h^3 + 2e^{2\alpha \beta h}/h^3 + e^{3\alpha \beta h}/h^3
)
```

Pairing axioms:

```
$TD = 5;
(U1[a, x, x] S-1[U-1[y, y, b]] // Expand // P1,-1) -
(S1[U1[a, x, x]] U-1[y, y, b] // Expand // P1,-1) // ExpandAB
0
((U1[a, x, x] U2[a, A[1], x] U-1[y, y, y, b, b, b] // m1,2->1 // Expand) // P1,-1) -
((U1[a, x, x] U2[a, A[1], x] U-1[y, y, y, b, b, b] // \Delta-1->-1,-2 // Expand) // P1,-1 // P2,-2)
0
((U1[a, x, x, x, x] U-2[y, B[1], b] U-1[y, y, y, b, b, b] // m-1,-2->-1 // Expand) // P1,-1) -
((U1[a, x, x, x, x] U-2[y, B[1], b] U-1[y, y, y, b, b, b] // \Delta-1->1,2 // Expand) // P1,-1 // P2,-2)
0
```

## The Double

```
DUForm[\mathcal{E}_] := \mathcal{E} //. U_i[fs___] U_j[gs___] /; i + j == 0 \wedge j > 0 \Rightarrow "DU"j[fs, gs];
DU_i[a] := U_i[] U_i[a]; DU_i[A[n_]] := U_i[] U_i[A[n]]; DU_i[x] := U_i[] U_i[x];
DU_i[y] := U_i[y] U_i[]; DU_i[b] := U_i[b] U_i[]; DU_i[B[n_]] := U_i[B[n]] U_i[];
DU_i[] := U_i[] U_i[];
```

```
DU1 /@ {a, A[1], x, y, b, B[-1]}
{U-1[] U1[a], U-1[] U1[A[1]], U-1[] U1[x], U-1[y] U1[], U-1[b] U1[], U-1[B[-1]] U1[]}
```

```

dmi,j→k[ε] := Module[{t1, t2, t3, h1, h2, h3},
  ε // Δ-j→t1,t2,t3 // St3 // Δi→h1,h2,h3 // Ph1,t1 // Ph3,t3 // mj,h2→k // m-i,t2→-k;
dΔi→j,k[ε] := ε // Δi→j,k // Δ-i→-j,-k;
dSi[ε] := Module[{h}, (ε // mi→h // Si // S-i) U-h[] Ui[] // Expand // dmh,i→i;

```

```

Module[{bas},
  bas = Join[{DU1[]}, DU1 /@ {a, A[n], x, y, B[m], b}, {β DU1[a] + α DU1[b]}];
  Table[If[f ** g != g ** f ∨ f === DU1[] ∨ g === DU1[], f ** g, "="], {f, bas}, {g, bas}] //
  ReplacePart[{1, 1} → "***"]
] // DUForm // MatrixForm

```

$$\left( \begin{array}{cccccc}
 * & & & & & \\
 DU_1[a] & = & DU_1[A[n]] & = & DU_1[x] & = & DU_1[y] \\
 DU_1[A[n]] & = & & = & -\alpha DU_1[x] + DU_1[a, x] & = & \alpha DU_1[y] + DU_1[y, a] \\
 DU_1[x] & DU_1[a, x] & DU_1[A[n], x] & = & e^{n\alpha\beta h} DU_1[A[n], x] & = & e^{-n\alpha\beta h} DU_1[y, A[n]] \\
 DU_1[y] & DU_1[y, a] & DU_1[y, A[n]] & = & = & = & -\frac{DU_1[A[1]]}{h} + \frac{DU_1[B[1]]}{h} + DU_1[y, x] \\
 DU_1[B[m]] & = & = & = & DU_1[y, x] & = & = \\
 DU_1[b] & = & = & = & DU_1[B[m], x] & = & e^{m\alpha\beta h} DU_1[y, B[m]] \\
 \beta DU_1[a] + \alpha DU_1[b] & = & = & = & DU_1[b, x] & = & -\beta DU_1[y] + DU_1[y, b] \\
 & & & & & & =
 \end{array} \right)$$

```

{DU1[A[1]] ** DU1[x], DU1[A[1]] ** DU1[y],
  DU1[x] ** DU1[B[1]], DU1[B[1]] ** DU1[y]} // DUForm
{eαβh DU1[A[1], x], e-αβh DU1[y, A[1]], eαβh DU1[B[1], x], eαβh DU1[y, B[1]]}

```

```

DU1[x] DU2[a] // dm1,2→1 // DUForm
DU1[a, x]

```

```

U-1[] U1[a] U-2[] U2[x] // dm1,2→1 // DUForm
-α DU1[x] + DU1[a, x]

```

```

U-1[] U1[a] U-2[b] U2[] // dm1,2→1 // DUForm
DU1[b, a]

```

```

{U-1[] U1[a] U-2[y] U2[] // dm2,1→1, U-1[] U1[a] U-2[y] U2[] // dm1,2→1} // DUForm
{DU1[y, a], α DU1[y] + DU1[y, a]}

```

```

(U-1[] U1[x] U-2[b] U2[] - U-1[b] U1[] U-2[] U2[x]) // dm1,2→1 // DUForm
-β DU1[x]

```

```

U-1[] U1[A[1]] U-2[y] U2[] // dm1,2→1
e-αβh U-1[y] U1[A[1]]

```

```

U-1[] U1[x] U-2[b] U2[] // dm1,2→1
-β U-1[] U1[x] + U-1[b] U1[x]

```

```

U-1[] U1[x] U-2[B[1]] U2[] // dm1,2→1
eαβh U-1[B[1]] U1[x]

```



$DU_1[x] DU_2[y] // dm_{1,2 \rightarrow 1} // DUForm$

$$- \frac{DU_1[A[1]]}{\hbar} + \frac{DU_1[B[1]]}{\hbar} + DU_1[y, x]$$

$z = U_{-1}[] U_1[x] U_{-2}[y] U_2[] U_{-3}[b] U_3[];$

$(z // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (z // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1}) // DUForm$

0

$\$TD = 5;$

$z = U_{-1}[y, b, b] U_1[A[2], a, x, x] U_{-2}[B[-1], y, y, b, b] U_2[a] U_{-3}[y, y, b] U_3[a, a, x, x, x];$

$(z // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (z // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1}) // DUForm$

0

$(\# \rightarrow DUForm[dS_1[DU_1[\#]]) \& /@ \{a, x, y, b\}$

$\{a \rightarrow -DU_1[a], x \rightarrow -e^{-\alpha \beta \hbar} DU_1[A[-1], x], y \rightarrow -DU_1[y, B[-1]], b \rightarrow -DU_1[b]\}$

$(DU_1[x] DU_2[y] // dm_{1,2 \rightarrow 1} // dS_1) - (DU_1[x] DU_2[y] // dS_1 // dS_2 // dm_{2,1 \rightarrow 1}) // ExpandAB$

0

Coassociativity

$(lhs = DU_1[\#] // d\Delta_{1 \rightarrow 1, 2} // d\Delta_{2 \rightarrow 2, 3}; rhs = DU_1[\#] // d\Delta_{1 \rightarrow 1, 3} // d\Delta_{1 \rightarrow 1, 2}; lhs == rhs) \& /@$   
 $\{a, x, y, b\}$

{True, True, True, True}

$d\Delta$  is algebra morphism

$z = U_{-1}[y, y, y, b, B[2]] U_1[x] U_{-2}[y, b, b, B[-3]] U_2[a, x];$

$(z // dm_{1,2 \rightarrow 1} // d\Delta_{1 \rightarrow 1, 2}) - (z // d\Delta_{2 \rightarrow 3, 4} // d\Delta_{1 \rightarrow 1, 2} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2})$

0

$dS$  is algebra anti-morphism

$dS$  is the convolution inverse of  $dm$ .

$z = U_{-1}[y, y, y, b, B[-3]] U_1[a, x, x];$

$\{z // d\Delta_{1 \rightarrow 1, 2} // dS_2 // dm_{1,2 \rightarrow 1}, z // d\Delta_{1 \rightarrow 1, 2} // dS_1 // dm_{1,2 \rightarrow 1}\}$

{0, 0}

## The R-Matrix

Quesne's formulas:

$$q = e^{\hbar \alpha \beta}; e_{q_-}[x_-] := \text{Exp}\left[\sum_{k=1}^{\$TD} \frac{(1-q)^k}{k(1-q^k)} x^k\right];$$

Table[Together@SeriesCoefficient[e<sub>ρ</sub>[x], {x, 0, n}], {n, 0, \$TD}]

$$\left\{1, 1, \frac{1}{1+\rho}, \frac{1}{(1+\rho)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)}, \right. \\ \left. 1/\left((1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)(1+\rho+\rho^2+\rho^3+\rho^4)\right)\right\}$$

```
R_{i_,j_} := O[U_{-i}[y -> y, b -> b] U_i[] U_{-j[]} U_j[a -> a, x -> x], e^{\hbar b a} e_q[\hbar x y]]
```

\$TD = 3; R\_{1,2} // DUForm

$$DU_1[] DU_2[] + \hbar DU_1[b] DU_2[a] + \hbar DU_1[y] DU_2[x] + \frac{1}{2} \hbar^2 DU_1[b, b] DU_2[a, a] + \hbar^2 DU_1[y, b] DU_2[a, x] + \\ \frac{1}{2} \hbar^2 DU_1[y, y] DU_2[x, x] - \frac{1}{4} \alpha \beta \hbar^3 DU_1[y, y] DU_2[x, x] + \frac{1}{6} \hbar^3 DU_1[b, b, b] DU_2[a, a, a] + \\ \frac{1}{2} \hbar^3 DU_1[y, b, b] DU_2[a, a, x] + \frac{1}{2} \hbar^3 DU_1[y, y, b] DU_2[a, x, x] + \frac{1}{6} \hbar^3 DU_1[y, y, y] DU_2[x, x, x]$$

\$TD = 5; {lhs = R\_{1,3} // dΔ\_{1→1,2}, rhs = R\_{2,3} \*\* R\_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm

0

\$TD = 5; {lhs = R\_{1,2} // dΔ\_{2→2,3}, rhs = R\_{1,2} \*\* R\_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm

0

\$TD = 5; Table[  
 {lhs = R\_{1,2} \*\* dΔ\_{1→2,1}[f], rhs = dΔ\_{1→1,2}[f] \*\* R\_{1,2}, lhs - rhs} // Last // ExpandAB // DUForm,  
 {f, {U\_{-1[]} U\_1[a], U\_{-1[]} U\_1[x], U\_{-1}[b] U\_1[], U\_{-1}[y] U\_1[]}}]  
 {0, 0, 0, 0}]

\$TD = 5;

{lhs = ExpandAB[R\_{1,2} \*\* R\_{1,3} \*\* R\_{2,3}],  
 rhs = ExpandAB[R\_{2,3} \*\* R\_{1,3} \*\* R\_{1,2}], Coefficient[lhs - rhs, \hbar^{TD}]} // Last

0

\$TD = 5; Si\_2[R\_{1,2}] \*\* R\_{1,2} // ExpandAB // DUForm

DU\_1[] DU\_2[]

### Cuaps (Unfinished)

Drinfeld element *u*.

```
u_{i_} := R_{1,2} // dS_1 // dm_{2,1→i}
```

`$TD = 2; u1 // ExpandAB // DUForm`

$$\begin{aligned} & DU_1[] - \beta \hbar DU_1[a] + \alpha \hbar DU_1[b] + \frac{1}{2} \beta^2 \hbar^2 DU_1[a, a] - \hbar DU_1[b, a] - \alpha \beta \hbar^2 DU_1[b, a] + \\ & \frac{1}{2} \alpha^2 \hbar^2 DU_1[b, b] - \hbar DU_1[y, x] + \alpha \beta \hbar^2 DU_1[y, x] + \beta \hbar^2 DU_1[b, a, a] - \alpha \hbar^2 DU_1[b, b, a] - \\ & 2 \alpha \hbar^2 DU_1[y, b, x] + \frac{1}{2} \hbar^2 DU_1[b, b, a, a] + \hbar^2 DU_1[y, b, a, x] + \frac{1}{2} \hbar^2 DU_1[y, y, x, x] \end{aligned}$$

Conjugation by the Drinfeld element implements the square of the antipode.

`$TD = 3; Table[DU1[f] ** u1 == u1 ** dS1[dS1[DU1[f]]] // ExpandAB, {f, {a, x, b, y}}]`  
`{True, True, True, True}`

Inverse of the Drinfeld element

$$u_{i-} := R_{1,2} // dS_2 // dS_2 // dm_{2,1 \rightarrow i}$$

`$TD = 3; ui1 ** u1 // ExpandAB`

$$U_{-1}[] U_1[]$$

Drinfeld commutes with its antipode and the product is central

`$TD = 3; u1 ** dS1[u1] - dS1[u1] ** u1 // ExpandAB`

$$0$$

`$TD = 3;`

`(u1 ** dS1[u1] ** DU1[#] - DU1[#] ** u1 ** dS1[u1]) & /@ {a, x, y, b} // ExpandAB`

$$\{0, 0, 0, 0\}$$

Multiplying the inverse and antipode of Drinfeld

`$TD = 3; ui1 ** dS1[u1] - U_{-1}[B[1]] U_1[A[-1]] // ExpandAB // DUForm`

$$0$$

This implies that  $dS[u] = uBA^{-1}$ . In other words,  $u dS[u] =$

$u^2 BA^{-1}$  so we can take the square root and call it  $v$ . This is the ribbon element.

$$v_{i-} := \left( 0[U_{-i}[b \rightarrow b] U_i[a \rightarrow a], \text{Exp}\left[\frac{\hbar}{2}(-\alpha b + \beta a)\right] \right) ** u_i$$

`$TD = 3; u1 ** dS1[u1] - v1 ** v1 // ExpandAB`

$$0$$

$v$  is supposed to be the Reidemeister 1 curl (with the appropriate cuaps/spinners added!). Note how it too is central.

`$TD = 3; (v1 ** DU1[#] - DU1[#] ** v1) & /@ {a, x, y, b} // ExpandAB`

$$\{0, 0, 0, 0\}$$

The rule for the cuaps is to add to a right-moving cup  $uv^{-1}$  and to a right – moving cap  $vu^{-1}$ . Note how

$$v u^{-1} = B^{1/2} A^{-1/2}$$

$$\$TD = 3; v_1 ** u_{i_1} - \mathcal{O} [U_{-1}[b \rightarrow b] U_1[a \rightarrow a], \text{Exp} \left[ \frac{\hbar}{2} (-\alpha b + \beta a) \right]] // \text{ExpandAB}$$

0

$$\text{cap}_{i_-} := \mathcal{O} [U_{-i}[b \rightarrow b] U_i[a \rightarrow a], \text{Exp} \left[ \frac{\hbar}{2} (-\alpha b + \beta a) \right]]$$

$$\text{cup}_{i_-} := \mathcal{O} [U_{-i}[b \rightarrow b] U_i[a \rightarrow a], \text{Exp} \left[ -\frac{\hbar}{2} (-\alpha b + \beta a) \right]]$$

Now let's do R2 and oppositely oriented Reidemeister 2:

$$\$TD = 3; dS_2[R_{1,2}] R_{3,4} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2} // \text{ExpandAB} // \text{DUForm}$$

DU<sub>1</sub> [] DU<sub>2</sub> []

$$\$TD = 3;$$

$$dS_2[R_{1,2}] R_{3,4} \text{cup}_5 \text{cap}_6 // dm_{1,3 \rightarrow 1} // dm_{4,5 \rightarrow 4} // dm_{4,2 \rightarrow 4} // dm_{4,6 \rightarrow 4} // \text{ExpandAB} // \text{DUForm}$$

DU<sub>1</sub> [] DU<sub>4</sub> []

Rotate a crossing:

$$\$TD = 3;$$

$$(\text{cup}_1 dS_5[R_{2,5}] \text{cap}_3 \text{cup}_4 \text{cap}_6 // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1} // dm_{4,5 \rightarrow 4} // dm_{4,6 \rightarrow 4}) - dS_4[R_{1,4}] // \text{ExpandAB}$$

0

Reidemeister 1 curls, the two positives agree with the two negatives and they are inverses and agree with  $v, v^{-1}$ .

$$\$TD = 2; \{$$

$$\begin{aligned} & (R_{1,2} \text{cap}_3 // dm_{1,3 \rightarrow 1} // dm_{1,2 \rightarrow 1}) - (R_{1,2} \text{cup}_3 // dm_{2,3 \rightarrow 2} // dm_{2,1 \rightarrow 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \rightarrow 1} // dm_{1,2 \rightarrow 1}) - (dS_2[R_{1,2}] \text{cap}_3 // dm_{2,3 \rightarrow 2} // dm_{2,1 \rightarrow 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \rightarrow 1} // dm_{1,2 \rightarrow 1}) ** (R_{1,2} \text{cap}_3 // dm_{1,3 \rightarrow 1} // dm_{1,2 \rightarrow 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \rightarrow 1} // dm_{1,2 \rightarrow 1}) - v_1 \} // \text{ExpandAB} // \text{DUForm} \end{aligned}$$

{0, 0, DU<sub>1</sub> [], 0}

## The Central Element

$$c_1 = \alpha U_{-1}[b] U_1[] + \beta U_{-1}[] U_1[a];$$

$$UB[c_1, \#] \& /@ \{U_{-1}[] U_1[a], U_{-1}[] U_1[x], U_{-1}[y] U_1[], U_{-1}[b] U_1[]\}$$

{0, 0, 0, 0}

## Commuting Exponentials

Commuting  $e^a$  with  $e^x$ :

$$\$TD = 5; \mathcal{O} [U_1[x \rightarrow x, a \rightarrow a], e^{\hbar (\mu x + \nu a)}] == \mathcal{O} [U_1[a \rightarrow a, x \rightarrow x], e^{\hbar (\mu e^{-\hbar \alpha \nu} x + \nu a)}]$$

True

Commuting  $e^b$  with  $e^y$ :

$$\text{\$TD} = 5; \text{\textcircled{O}} [U_1[b \rightarrow b, y \rightarrow y], e^{\hbar(\mu y + \nu b)}] == \text{\textcircled{O}} [U_1[y \rightarrow y, b \rightarrow b], e^{\hbar(\mu e^{-\hbar\beta\nu} y + \nu b)}]$$

True

The co-product of  $e^a$ :

$$\text{\$TD} = 5; \Delta_{1 \rightarrow 1, 2} [\text{\textcircled{O}} [U_1[a \rightarrow a], e^{\hbar\mu a}]] == \text{\textcircled{O}} [U_1[a_1 \rightarrow a] U_2[a_2 \rightarrow a], e^{\hbar\mu a_1} e^{\hbar\mu a_2}]$$

True

The co-product of  $e_q^x$ :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \Delta_{1 \rightarrow 1, 2} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[a_1 \rightarrow a, x_1 \rightarrow x] U_2[x_2 \rightarrow x], e_q[\hbar\mu e^{-\hbar\beta a_1} x_2] e_q[\hbar\mu x_1]]$$

True

The triple co-product of  $e_q^x$ :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \Delta_{1 \rightarrow 1, 2, 3} // \text{ExpandAB}) - \text{\textcircled{O}} [U_1[a_1 \rightarrow a, x_1 \rightarrow x] U_2[a_2 \rightarrow a, x_2 \rightarrow x] U_3[x_3 \rightarrow x], e_q[\hbar\mu e^{-\hbar\beta(a_1+a_2)} x_3] e_q[\hbar\mu e^{-\hbar\beta a_1} x_2] e_q[\hbar\mu x_1]]$$

0

The co-product of  $e_q^y$ :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[y \rightarrow y], e_q[\hbar\nu y]] // \Delta_{1 \rightarrow 1, 2} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[y_1 \rightarrow y] U_2[y_2 \rightarrow y, b_2 \rightarrow b], e_q[\hbar\nu y_2] e_q[\hbar\nu e^{-\hbar\alpha b_2} y_1]]$$

True

The triple co-product of  $e_q^y$ :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[y \rightarrow y], e_q[\hbar\nu y]] // \Delta_{1 \rightarrow 1, 2, 3} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[y_1 \rightarrow y] U_2[y_2 \rightarrow y, b_2 \rightarrow b] U_3[y_3 \rightarrow y, b_3 \rightarrow b], e_q[\hbar\nu y_3] e_q[\hbar\nu e^{-\hbar\alpha b_3} y_2] e_q[\hbar\nu e^{-\hbar\alpha(b_2+b_3)} y_1]]$$

True

The inverse-antipode on  $e_q^x$ :

$$\text{\$TD} = 8; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \text{Si}_1 // \text{ExpandAB}) - \left( \text{\textcircled{O}} [U_1[a \rightarrow a, x \rightarrow x], \text{Sum} \left[ \frac{(-\hbar\mu)^k q^{-k(k+1)/2} e^{\hbar\beta k a} x^k}{\text{QFactorial}[k, q]}, \{k, 0, \text{\$TD}\} \right] \right)$$

0

Pairing  $e_q^x$  with  $e_q^y$ :

```

$TD = 5;
$TD *= 2;
lhs = O[U_{-1}[y → y], e_q[ħ v y]] O[U_1[x → x], e_q[ħ μ x]] // Expand // P_{1,-1} // ExpandAB;
($TD /= 2; e_q[ħ μ v] - lhs // ExpandAB)
0

```

## Computing $e^{\mu x} y e^{-\mu x}$ .

Following BBS:VanDerVeen-170620-212242, though with explicit  $\hbar$  put into  $\mu$ ,  $v$ .

```

φ_0[x_] := e^x;
φ_k[x_] := Module[{t}, Simplify[ $\frac{e^t - \text{Normal@Series}[e^t, \{t, 0, k - 1\}]}{t^k}$  /. t → x]];

```

```
Table[φ_k[x], {k, 0, 3}]
```

```
{e^x,  $\frac{-1 + e^x}{x}$ ,  $\frac{-1 + e^x - x}{x^2}$ ,  $-\frac{2 - 2e^x + 2x + x^2}{2x^3}$ }
```

\$TD = 5;

GS = T μ (A<sup>-1</sup> φ<sub>1</sub> [ħ μ (q - 1) x] - 1) - μ (A φ<sub>1</sub> [ħ μ (q<sup>-1</sup> - 1) x] - 1) /. {A -> e<sup>-ħβa</sup>};

(G = O[U<sub>-1</sub>[ ] U<sub>1</sub>[x -> x, a -> a], GS]) // DUForm

{lhs = O[U<sub>-1</sub>[ ] U<sub>1</sub>[x -> x], e<sup>ħμx</sup>] \*\* DU<sub>1</sub>[y] \*\* O[U<sub>-1</sub>[ ] U<sub>1</sub>[x -> x], e<sup>-ħμx</sup>],

rhs = Expand[DU<sub>1</sub>[y] + μ (T - 1) DU<sub>1</sub>[ ] + G],

lhs - rhs /. T \* ε -> DU<sub>1</sub>[A[1]] \*\* DU<sub>1</sub>[B[1]] \*\* ε} // Last // ExpandAB // DUForm

$$\begin{aligned} & \beta \mu \hbar \text{DU}_1[a] + T \beta \mu \hbar \text{DU}_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] + \frac{1}{12} \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] + \frac{1}{12} T \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] + \\ & \frac{1}{48} \alpha^4 \beta^4 \mu^2 \hbar^5 \text{DU}_1[x] - \frac{1}{48} T \alpha^4 \beta^4 \mu^2 \hbar^5 \text{DU}_1[x] - \frac{1}{2} \beta^2 \mu \hbar^2 \text{DU}_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 \text{DU}_1[a, a] - \\ & \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] - \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \\ & \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \frac{1}{12} \alpha^3 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, x] + \frac{1}{12} T \alpha^3 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, x] - \\ & \frac{1}{6} \alpha^2 \beta^2 \mu^3 \hbar^4 \text{DU}_1[x, x] + \frac{1}{6} T \alpha^2 \beta^2 \mu^3 \hbar^4 \text{DU}_1[x, x] - \frac{1}{6} \alpha^3 \beta^3 \mu^3 \hbar^5 \text{DU}_1[x, x] - \\ & \frac{1}{6} T \alpha^3 \beta^3 \mu^3 \hbar^5 \text{DU}_1[x, x] + \frac{1}{6} \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \frac{1}{6} T \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \\ & \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{8} \alpha^2 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, x] - \\ & \frac{1}{8} T \alpha^2 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, x] + \frac{1}{6} \alpha^2 \beta^3 \mu^3 \hbar^5 \text{DU}_1[a, x, x] + \frac{1}{6} T \alpha^2 \beta^3 \mu^3 \hbar^5 \text{DU}_1[a, x, x] - \\ & \frac{1}{24} \beta^4 \mu \hbar^4 \text{DU}_1[a, a, a, a] + \frac{1}{24} T \beta^4 \mu \hbar^4 \text{DU}_1[a, a, a, a] - \frac{1}{12} \alpha \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, a, x] + \\ & \frac{1}{12} T \alpha \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, a, x] + \frac{1}{120} \beta^5 \mu \hbar^5 \text{DU}_1[a, a, a, a, a] + \frac{1}{120} T \beta^5 \mu \hbar^5 \text{DU}_1[a, a, a, a, a] \end{aligned}$$

0

K = 3; \$TD = 5;

GS1 =  $\mu T (A^{-1} \phi_1 [\hbar \mu (q - 1) x] - 1) - \mu (A \phi_1 [\hbar \mu (q^{-1} - 1) x] - 1) /. \{A \rightarrow e^{-\hbar \beta a}\};$

(G1 =  $\mathcal{O}[U_{-1}[] U_1[x \rightarrow x], a \rightarrow a]$ , GS1) // DUForm

Collect[ExpandAB[

Expand[ $DU_1[y] + (T - 1) \mu DU_1[] + G1 - \mathcal{O}[U_{-1}[] U_1[x \rightarrow x], e^{\hbar \mu x}] ** DU_1[y] **$

$\mathcal{O}[U_{-1}[] U_1[x \rightarrow x], e^{-\hbar \mu x}] /. T * \mathcal{E}_- \rightarrow \mathcal{E} ** DU_1[A[1]] ** DU_1[B[1]]$

],

$\beta]$

$$\begin{aligned} & \beta \mu \hbar DU_1[a] + T \beta \mu \hbar DU_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 DU_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 DU_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 DU_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 DU_1[x] + \frac{1}{12} \alpha^3 \beta^3 \mu^2 \hbar^4 DU_1[x] + \frac{1}{12} T \alpha^3 \beta^3 \mu^2 \hbar^4 DU_1[x] + \\ & \frac{1}{48} \alpha^4 \beta^4 \mu^2 \hbar^5 DU_1[x] - \frac{1}{48} T \alpha^4 \beta^4 \mu^2 \hbar^5 DU_1[x] - \frac{1}{2} \beta^2 \mu \hbar^2 DU_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 DU_1[a, a] - \\ & \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 DU_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 DU_1[a, x] - \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 DU_1[a, x] - \\ & \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 DU_1[a, x] - \frac{1}{12} \alpha^3 \beta^4 \mu^2 \hbar^5 DU_1[a, x] + \frac{1}{12} T \alpha^3 \beta^4 \mu^2 \hbar^5 DU_1[a, x] - \\ & \frac{1}{6} \alpha^2 \beta^2 \mu^3 \hbar^4 DU_1[x, x] + \frac{1}{6} T \alpha^2 \beta^2 \mu^3 \hbar^4 DU_1[x, x] - \frac{1}{6} \alpha^3 \beta^3 \mu^3 \hbar^5 DU_1[x, x] - \\ & \frac{1}{6} T \alpha^3 \beta^3 \mu^3 \hbar^5 DU_1[x, x] + \frac{1}{6} \beta^3 \mu \hbar^3 DU_1[a, a, a] + \frac{1}{6} T \beta^3 \mu \hbar^3 DU_1[a, a, a] + \\ & \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 DU_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 DU_1[a, a, x] + \frac{1}{8} \alpha^2 \beta^4 \mu^2 \hbar^5 DU_1[a, a, x] - \\ & \frac{1}{8} T \alpha^2 \beta^4 \mu^2 \hbar^5 DU_1[a, a, x] + \frac{1}{6} \alpha^2 \beta^3 \mu^3 \hbar^5 DU_1[a, x, x] + \frac{1}{6} T \alpha^2 \beta^3 \mu^3 \hbar^5 DU_1[a, x, x] - \\ & \frac{1}{24} \beta^4 \mu \hbar^4 DU_1[a, a, a, a] + \frac{1}{24} T \beta^4 \mu \hbar^4 DU_1[a, a, a, a] - \frac{1}{12} \alpha \beta^4 \mu^2 \hbar^5 DU_1[a, a, a, x] + \\ & \frac{1}{12} T \alpha \beta^4 \mu^2 \hbar^5 DU_1[a, a, a, x] + \frac{1}{120} \beta^5 \mu \hbar^5 DU_1[a, a, a, a, a] + \frac{1}{120} T \beta^5 \mu \hbar^5 DU_1[a, a, a, a, a] \end{aligned}$$

0



**K = 3; \$TD = 5;**

**GS2 = Collect[Normal@Series[GS1, {β, 0, K}], β, Factor];**

**(G2 = O[U<sub>-1</sub>[ ] U<sub>1</sub>[x → x, a → a], GS2]) // DUForm**

**Collect[ExpandAB[**

**Expand[DU<sub>1</sub>[y] + (T - 1) μ DU<sub>1</sub>[ ] + G2 - O[U<sub>-1</sub>[ ] U<sub>1</sub>[x → x], e<sup>ħ μ x</sup>] \*\* DU<sub>1</sub>[y] \*\***  
**O[U<sub>-1</sub>[ ] U<sub>1</sub>[x → x], e<sup>-ħ μ x</sup>]] /. T \* ε<sub>-</sub> => ε \*\* DU<sub>1</sub>[A[1]] \*\* DU<sub>1</sub>[B[1]]**

**],**  
**β]**

$$\begin{aligned} & \beta \mu \hbar \text{DU}_1[a] + T \beta \mu \hbar \text{DU}_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{5}{24} \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] - \frac{5}{24} T \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] - \\ & \frac{1}{2} \beta^2 \mu \hbar^2 \text{DU}_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 \text{DU}_1[a, a] - \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] - \\ & \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] + \frac{1}{6} \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \\ & \frac{1}{6} T \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] \\ & \frac{1}{6} \alpha^3 \beta^2 \mu^3 \hbar^5 \text{U}_{-1}[b] \text{U}_1[x, x] + \\ & \beta^3 \left( -\frac{7}{12} \alpha^3 \mu^2 \hbar^4 \text{U}_{-1}[ ] \text{U}_1[x] + \frac{7}{24} \alpha^4 \mu^2 \hbar^5 \text{U}_{-1}[b] \text{U}_1[x] - \frac{1}{6} \alpha^2 \mu^3 \hbar^5 \text{U}_{-1}[ ] \text{U}_1[a, x, x] \right) + \\ & \beta^4 \left( -\frac{7}{24} \alpha^4 \mu^2 \hbar^5 \text{U}_{-1}[ ] \text{U}_1[x] + \frac{7}{24} \alpha^3 \mu^2 \hbar^5 \text{U}_{-1}[ ] \text{U}_1[a, x] + \frac{1}{24} \alpha \mu \hbar^5 \text{U}_{-1}[b] \text{U}_1[a, a, a, a] \right) + \\ & \frac{1}{40} \beta^5 \mu \hbar^5 \text{U}_{-1}[ ] \text{U}_1[a, a, a, a, a] \end{aligned}$$

**Clear[ħ]**

**K = 3; \$TD = 5;**

**GS1 = μ T (A<sup>-1</sup> φ<sub>1</sub>[ħ μ (q - 1) x] - 1) - μ (A φ<sub>1</sub>[ħ μ (q<sup>-1</sup> - 1) x] - 1) /. {A → e<sup>-ħ β a</sup>}**

$$-\mu \left( -1 + \frac{e^{-a \beta \hbar} (-1 + e^{(-1 + e^{-a \beta \hbar}) x \mu \hbar})}{(-1 + e^{-\alpha \beta \hbar}) x \mu \hbar} \right) + T \mu \left( -1 + \frac{e^{a \beta \hbar} (-1 + e^{(-1 + e^{\alpha \beta \hbar}) x \mu \hbar})}{(-1 + e^{\alpha \beta \hbar}) x \mu \hbar} \right)$$

**K = 3;**

**GS2 = Collect[Normal@(GS1 + O[β]<sup>K+1</sup>), β, Factor]**

$$\frac{1}{2} (1 + T) \beta \mu \hbar (2 a + x \alpha \mu \hbar) + \frac{1}{4} (-1 + T) \beta^2 \mu \hbar^2 (2 a^2 + 2 a x \alpha \mu \hbar + x \alpha^2 \mu \hbar) +$$

$$\frac{1}{24} (1 + T) \beta^3 \mu \hbar^3 (4 a^3 + 6 a^2 x \alpha \mu \hbar + 6 a x \alpha^2 \mu \hbar - 5 x \alpha^3 \mu \hbar)$$

$\mathcal{O}$  [GS1 - GS2]

$$\frac{1}{24} \left( -a^4 \beta^4 \mu + a^4 \tau \beta^4 \mu + 7 x \alpha^3 \beta^3 \mu^2 + 7 \tau x \alpha^3 \beta^3 \mu^2 - 4 x^2 \alpha^2 \beta^2 \mu^3 + 4 \tau x^2 \alpha^2 \beta^2 \mu^3 \right) \hbar^4 + \left( \frac{1}{240} \mu \left( 2 a^5 \beta^5 - 20 a^3 x \alpha \beta^4 \mu - 30 a^2 x \alpha^2 \beta^4 \mu - 20 a x \alpha^3 \beta^4 \mu - 5 x \alpha^4 \beta^4 \mu + 40 a x^2 \alpha^2 \beta^3 \mu^2 + 40 x^2 \alpha^3 \beta^3 \mu^2 \right) + \frac{1}{240} \tau \mu \left( 2 a^5 \beta^5 + 20 a^3 x \alpha \beta^4 \mu + 30 a^2 x \alpha^2 \beta^4 \mu + 20 a x \alpha^3 \beta^4 \mu + 5 x \alpha^4 \beta^4 \mu + 40 a x^2 \alpha^2 \beta^3 \mu^2 + 40 x^2 \alpha^3 \beta^3 \mu^2 \right) \right) \hbar^5$$

Series[Normal@ $\mathcal{O}$  [GS1 - GS2], { $\beta$ ,  $\theta$ , K}]

$$\frac{1}{6} \left( -x^2 \alpha^2 \mu^3 \hbar^4 + \tau x^2 \alpha^2 \mu^3 \hbar^4 \right) \beta^2 + \frac{1}{24} \left( 7 x \alpha^3 \mu^2 \hbar^4 + 7 \tau x \alpha^3 \mu^2 \hbar^4 + 4 a x^2 \alpha^2 \mu^3 \hbar^5 + 4 a \tau x^2 \alpha^2 \mu^3 \hbar^5 + 4 x^2 \alpha^3 \mu^3 \hbar^5 + 4 \tau x^2 \alpha^3 \mu^3 \hbar^5 \right) \beta^3 + \mathcal{O}[\beta]^4$$

Normal@ $\mathcal{O}$  [GS1 - GS2] +  $\mathcal{O}[\beta]^{K+1}$

$$\frac{1}{6} \left( -x^2 \alpha^2 \mu^3 \hbar^4 + \tau x^2 \alpha^2 \mu^3 \hbar^4 \right) \beta^2 + \frac{1}{24} \left( 7 x \alpha^3 \mu^2 \hbar^4 + 7 \tau x \alpha^3 \mu^2 \hbar^4 + 4 a x^2 \alpha^2 \mu^3 \hbar^5 + 4 a \tau x^2 \alpha^2 \mu^3 \hbar^5 + 4 x^2 \alpha^3 \mu^3 \hbar^5 + 4 \tau x^2 \alpha^3 \mu^3 \hbar^5 \right) \beta^3 + \mathcal{O}[\beta]^4$$

Series[GS1 - GS2, { $\beta$ ,  $\theta$ , K}]

$$\mathcal{O}[\beta]^4$$