

Pensieve header: Figuring out the double of the 2D pencil; older versions under "MetaDouble-**".

Issues:

1. Improve DUForm.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-06"];
```

The "degree carrier" is \hbar , and all "coupling constants" are proportional to it.

```
$TD = 3;  $\hbar$  /:  $\hbar^{d-}$  /;  $d > $TD := 0;$ 
```

The 2D Lie BiAlgebra Pencil

We hope to stick to $A = e^{\hbar\beta a}$ and to $B = e^{\hbar\alpha b}$, where $[a, x] = \alpha x$ and $[b, y] = -\beta y$.

Also, $\Delta_{12}(a, A, x, b, B, y) = (a_1 + a_2, A_1 A_2, x_1 + A_1 x_2, b_1 + b_2, B_1 B_2, y_1 B_2 + y_2)$.

Also, (a, x) and $\hbar(b, y)$ are dual bases.

```
AlgebraAtom = a | A[_] | x | b | B[_] | y;
$PBWRule = {A[_] -> 1, a -> 2, x -> 3, y -> 4, B[_] -> 5, b -> 6};
```

```
B[a, x] =  $\alpha x$ ; B[x, A[n_]] = (e $\hbar\alpha\beta$  - 1) U[A[n], x]; B[a, A[_]] = 0;
B[y, b] =  $\beta y$ ; B[B[n_], y] = (e $\hbar\alpha\beta$  - 1) U[y, B[n]]; B[b, B[_]] = 0;
 $\Delta$ [a] = U1[a] U2[] + U1[] U2[a];  $\Delta$ [A[n_]] := U1[A[n]] U2[A[n]];
 $\Delta$ [x] = U1[x] U2[] + U1[A[1]] U2[x];
 $\Delta$ [b] = U1[b] U2[] + U1[] U2[b];  $\Delta$ [B[n_]] := U1[B[n]] U2[B[n]];
 $\Delta$ [y] = U1[y] U2[B[1]] + U1[] U2[y];
S[a] = -a; S[A[n_]] := A[-n]; S[x] = -U[A[-1], x];
S[b] = -b; S[B[n_]] := B[-n]; S[y] = -U[y, B[-1]];
Si[a] = -a; Si[A[n_]] := A[-n]; Si[x] = -U[x, A[-1]];
Si[b] = -b; Si[B[n_]] := B[-n]; Si[y] = -U[B[-1], y];
(* This extra line is annoying *)
```

```
ExpandAB[ $\mathcal{E}$ ] := Expand@Normal@Series[ $\mathcal{E}$  //. {
  c_. Ui[ $\lambda$ _, A[n_],  $\rho$ _] =>
  Expand[c Sum[ $\frac{(-1)^d \hbar^d \beta^d n^d}{d!}$  Ui[ $\lambda$ , Sequence@@Table[a, {d}],  $\rho$ ], {d, 0, $TD}]],
  c_. Ui[ $\lambda$ _, B[n_],  $\rho$ _] => Expand[
  c Sum[ $\frac{(-1)^d \hbar^d \alpha^d n^d}{d!}$  Ui[ $\lambda$ , Sequence@@Table[b, {d}],  $\rho$ ], {d, 0, $TD}]]
},
{ $\hbar$ ,
0,
$TD}]
```

UEA with provisional modification

This section is based on `pensieve://Projects/UEA/`.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i[] := U_i[]; UU_i[1] := U_i[];
UU_i[x_[n_]^p_] := U_i[x[n p]];
UU_i[x_^p_] := UU_i@@Table[x, {p}];
UU_i[ε_] := ε /. {
  U[xs_] => UU_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i[x_, xs_] := UU_t1[x] UU_t2[xs] // Expand // m_t1,t2->i;
USimp[ε_] := Collect[ε, Times[U[___] ..], Expand];
USimp[ε_] := Expand[ε];
```

```
m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_i->j_[ε_] := ε /. U_i -> U_j;
```

```
m_i,j->k_[c_. U_i[x___] U_j[]] := c U_k[x];
m_i,j->k_[c_. U_i[] U_j[y___]] := c U_k[y];
m_i,j->k_[c_. U_i[xx___, x_[n1_]] U_j[x_[n2_], yy___]] :=
  USimp[c If[TrueQ[n1 + n2 == 0], U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j->k];
m_i,j->k_[c_. U_i[xx___, x_] U_j[y_, yy___]] := If[x ≤ y,
  c U_k[xx, x, y, yy],
  ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]])) // Expand // m_i,j->i) U_j[yy] // Expand // m_i,j->k)
  c // USimp
];
```

```
Supp[ε_] := Union@Cases[{ε}, U_i[___] -> i, ∞];
```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
  {Sx = Supp[x], Sy = Supp[y], is, σ, z},
  If[MatchQ[Sx ∪ Sy, {_Integer ...}] && Min[Sx ∪ Sy] < 0,
    is = Abs[Sx] ∩ Abs[Sy];
    z = x; Do[z = m-i→-σ@i[mi→σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dmσ@i, i→i[z], {i, is}];
    z,
    (* else *) is = Sx ∩ Sy;
    z = x; Do[z = mi→σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = mσ@i, i→i[z], {i, is}];
    z
  ]
];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[func_] := Normal@Series[func, {ħ, 0, $TD}];
O[specs_, func_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U_[u___] => Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[O[func], vars] /. (ps_ -> c_) => c (
    specs /. MapThread[({#1 -> _} => #3^#2) &, {vars, ps, elems}] /. Ui -> UUi
  )]
]

```

The 2D Lie BiAlgebra Pencil, Testing

```

O[U1[a -> a], eħβa]
U1[ ] + β ħ U1[a] +  $\frac{1}{2} \beta^2 \hbar^2 U_1[a, a] + \frac{1}{6} \beta^3 \hbar^3 U_1[a, a, a]$ 

USimp@With[{An = O[U1[a -> a], e-nħβa]}, UB[U1[x], An] - O[enħαβ - 1] An ** U1[x]]
0

B[x, A[3]]
(-1 + e3αβħ) U[A[3], x]

$TD = 6;
USimp@With[{Bn = O[U1[b -> b], e-nħαb]}, UB[Bn, U1[y]] - O[enħαβ - 1] U1[y] ** Bn]
0

z = U1[a, A[2], x, x, x] U2[a, a, x] U3[a, a, A[-3], x];
(z // m1,2→1 // m1,3→1) - (z // m2,3→2 // m1,2→1)
0

```

```
z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];
(z // m1,2→1 // m1,3→1) - (z // m2,3→2 // m1,2→1)
0
```

The Co-Product and Co-Associativity

```
Δi→j-,k- [ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[f-, fs-] :=
    (USimp[Δ[f] /. {U1 → Uj, U2 → Uk}) Δi→tj,tk[Ui[fs]] // mj,tj→j // mk,tk→k
}];
Δi→j-,k-,l- [ε-] := ε // Δi→j,k // Δk→l
```

```
Δ1→1,2[U1[#]] & /@ {a, A[7], x, y, b, B[-3]}
{U1[a] U2[] + U1[] U2[a], U1[A[7]] U2[A[7]], U1[x] U2[] + U1[A[1]] U2[x],
 U1[] U2[y] + U1[y] U2[B[1]], U1[b] U2[] + U1[] U2[b], U1[B[-3]] U2[B[-3]]}

{lhs = U1[x] // Δ1→1,2 // Δ2→2,3, rhs = U1[x] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
{U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x],
 U1[x] U2[] U3[] + U1[A[1]] U2[x] U3[] + U1[A[1]] U2[A[1]] U3[x], True}

U1[y] // Δ1→1,2
U1[] U2[y] + U1[y] U2[B[1]]

{lhs = U1[y] // Δ1→1,2 // Δ2→2,3, rhs = U1[y] // Δ1→1,3 // Δ1→1,2, lhs == rhs}
{U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]],
 U1[] U2[] U3[y] + U1[] U2[y] U3[B[1]] + U1[y] U2[B[1]] U3[B[1]], True}

z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
0

z = U1[y, y, y, b, B[2]] U2[y, b, b, B[-3]];
(z // m1,2→1 // Δ1→1,2) - (z // Δ2→3,4 // Δ1→1,2 // m1,3→1 // m2,4→2)
0
```

The Antipode

```
Si- [ε-] := Module[{ti}, USimp[
  ε /. Ui[x-, xs-] := mti,i→i[Expand[UUi[S[x]] S ti[U ti[xs]]]]
]];
Si- [ε-] := Module[{ti}, USimp[
  ε /. Ui[x-, xs-] := mti,i→i[Expand[UUi[Si[x]] Si ti[U ti[xs]]]]
]];

```

```

{U1[x] // S1 // Si, U1[y] // S1 // Si}
{e^{\alpha\beta\hbar} U1[x], e^{\alpha\beta\hbar} U1[y]}

{U1[x] // S1 // Si1, U1[y] // S1 // Si1}
{U1[x], U1[y]}

{z = U1[]; (z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}), z = U1[]; (z // \Delta_{1\to 2} // S2 // m_{1,2\to 1})}
{U1[], U1[]}

z = U1[a, A[3], x, x];
{z // \Delta_{1\to 2} // S2 // m_{1,2\to 1}, z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}}
{0, 0}

z = U1[y, y, y, b, B[-3]];
{z // \Delta_{1\to 2} // S2 // m_{1,2\to 1}, z // \Delta_{1\to 2} // S1 // m_{1,2\to 1}}
{0, 0}

{z = U1[a, A[2], x, x, x] U2[a, a, A[-3], x]; (z // m_{1,2\to 1} // S1) - (z // S1 // S2 // m_{2,1\to 1}),
z = U1[y, y, y, b, B[2]] U2[y, b, b, b, B[6]]; (z // m_{1,2\to 1} // S1) - (z // S1 // S2 // m_{2,1\to 1})}
{0, 0}

{z = U1[a, A[2], x, x, x]; (z // S1 // \Delta_{1\to 2,1}) - (z // \Delta_{1\to 2} // S1 // S2),
z = U1[y, y, y, b, B[2]]; (z // S1 // \Delta_{1\to 2,1}) - (z // \Delta_{1\to 2} // S1 // S2)}
}
{0, 0}

```

The Pairing

$$\begin{aligned}
 &P[U[], U[B[_]]] = P[U[A[_]], U[]] = 1; \\
 &P[U[], U[[_]]] = P[U[[_]], U[]] = 0; \\
 &\left(\begin{array}{lll}
 P[U[a], U[b]] = \hbar^{-1} & P[U[a], U[B[n_]]] = -n\alpha & P[U[a], U[y]] = 0 \\
 P[U[A[n_]], U[b]] = -n\beta & P[U[A[n_]], U[B[m_]]] = e^{nm\hbar\alpha\beta} & P[U[A[_]], U[y]] = 0 \\
 P[U[x], U[b]] = 0 & P[U[x], U[B[_]]] = 0 & P[U[x], U[y]] = \hbar^{-1}
 \end{array} \right);
 \end{aligned}$$

$$\begin{aligned}
 &P[U[], U[]] = 1; \\
 &P_{i,j}[\mathcal{E}] := \text{USimp}[\mathcal{E} /. U_i[xs_]\ U_j[ys_]\ \to P[U[xs], U[ys]]];
 \end{aligned}$$

The pairing sequence: (one,one) (above), (many,one), (many,many).

$$\begin{aligned}
 &P[U[x_], xs_], U[y_]] := P[U[x, xs], U[y]] = \\
 &\quad \text{Module}[\{i, j, k, l\}, \text{USimp}[U_i[x] \ U_j[xs] \ \Delta_{k\to l,1}[U_k[y]]] // P_{i,k} // P_{j,l}]; \\
 &P[U[xs_], U[y_, ys_]] := P[U[xs], U[y, ys]] = \\
 &\quad \text{Module}[\{i, j, k, l\}, \text{USimp}[\Delta_{i\to j,1}[U_i[xs]] \ U_k[y] \ U_l[ys]] // P_{i,k} // P_{j,l}];
 \end{aligned}$$

```

z = Ui[a] Uj[x] Uk[y];
{mi,j→i[z] - mj,i→i[z], Δk→k,1[z] - Δk→1,k[z]}
{α Ui[x] Uk[y], -Ui[a] Uj[x] Uk[y] U1[ ] +
  Ui[a] Uj[x] Uk[ ] U1[y] - Ui[a] Uj[x] Uk[B[1]] U1[y] + Ui[a] Uj[x] Uk[y] U1[B[1]]}

```

```

Table[z = Ui[xi] Uj[xj] Uk[yk];
{(mi,j→i[z] - mj,i→i[z]) // Pi,k, (Δk→k,1[z] - Δk→1,k[z]) // Pi,k // Pj,1},
{xi, {a, x}}, {xj, {a, x}}, {yk, {b, y}}]
{{{(0, 0), (0, 0)}, {(0, 0), {α/h, α/h}}}, {{{(0, 0), {-α/h, -α/h}}, {(0, 0), (0, 0)}}}

```

```

Table[z = Ui[xi] Uk[yk] U1[y1];
{(Δi→i,j[z] - Δi→j,i[z]) // Pi,k // Pj,1, (mk,1→k[z] - m1,k→k[z]) // Pi,k},
{xi, {a, x}}, {yk, {b, y}}, {y1, {b, y}}]
{{{(0, 0), (0, 0)}, {(0, 0), (0, 0)}}, {{{(0, 0), {-β/h, -β/h}}, {{β/h, β/h}, (0, 0)}}}

```

```

lhs = Factor@Table[hn P[U@@Table[x, {n}], U@@Table[y, {n}]], {n, $TD = 7}]
{1, 1 + eαβh, (1 + eαβh) (1 + eαβh + e2αβh), (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh),
(1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh),
(1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2 (1 + eαβh + e2αβh + e3αβh + e4αβh),
(1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2
(1 + eαβh + e2αβh + e3αβh + e4αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh + e5αβh + e6αβh)}

```

```

rhs = Simplify@FunctionExpand@Table[QFactorial[n, ehαβ], {n, $TD = 7}]
{1, 1 + eαβh, (1 + eαβh) (1 + eαβh + e2αβh), (1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh),
(1 + eαβh)2 (1 + e2αβh) (1 + eαβh + e2αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh),
(1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2 (1 + eαβh + e2αβh + e3αβh + e4αβh),
(1 + eαβh)3 (1 + e2αβh) (1 - eαβh + e2αβh) (1 + eαβh + e2αβh)2
(1 + eαβh + e2αβh + e3αβh + e4αβh) (1 + eαβh + e2αβh + e3αβh + e4αβh + e5αβh + e6αβh)}

```

```

MapThread[Equal, {lhs, rhs}]
{True, True, True, True, True, True, True}

```

```

P[U[a, a, a, a, a], U[b, b, b, b, b]]

```

$$\frac{120}{h^5}$$

```
Table[P[z1, z2],
  {z1, {U[], U[a], U[x], U[a, a], U[a, x], U[x, x],
    U[a, a, a], U[a, a, x], U[a, x, x], U[x, x, x]}}, {z2, {U[], U[b], U[y],
    U[b, b], U[y, b], U[y, y], U[b, b, b], U[y, b, b], U[y, y, b], U[y, y, y]}
] //
MatrixForm
(
  1 0 0 0 0 0 0 0 0 0
  0 1/h 0 0 0 0 0 0 0 0
  0 0 1/h 0 0 0 0 0 0 0
  0 0 0 2/h^2 0 0 0 0 0 0
  0 0 0 0 1/h^2 0 0 0 0 0
  0 0 0 0 0 1/h^2 + e^{\alpha \beta h}/h^2 0 0 0 0
  0 0 0 0 0 0 6/h^3 0 0 0
  0 0 0 0 0 0 0 2/h^3 0 0
  0 0 0 0 0 0 0 0 1/h^3 + e^{\alpha \beta h}/h^3 0
  0 0 0 0 0 0 0 0 0 1/h^3 + 2e^{\alpha \beta h}/h^3 + 2e^{2\alpha \beta h}/h^3 + e^{3\alpha \beta h}/h^3
)
```

Pairing axioms:

```
$TD = 5;
(U1[a, x, x] S-1[U-1[y, y, b]] // Expand // P1,-1) -
(S1[U1[a, x, x]] U-1[y, y, b] // Expand // P1,-1) // ExpandAB
0
((U1[a, x, x] U2[a, A[1], x] U-1[y, y, y, b, b, b] // m1,2->1 // Expand) // P1,-1) -
((U1[a, x, x] U2[a, A[1], x] U-1[y, y, y, b, b, b] // \Delta-1->-1,-2 // Expand) // P1,-1 // P2,-2)
0
((U1[a, x, x, x, x] U-2[y, B[1], b] U-1[y, y, y, b, b, b] // m-1,-2->-1 // Expand) // P1,-1) -
((U1[a, x, x, x, x] U-2[y, B[1], b] U-1[y, y, y, b, b, b] // \Delta-1->1,2 // Expand) // P1,-1 // P2,-2)
0
```

The Double

```
DUForm[\mathcal{E}_] := \mathcal{E} //. U_i[fs___] U_j[gs___] /; i + j == 0 \wedge j > 0 => "DU"j[fs, gs];
DU_i[a] := U_i[] U_i[a]; DU_i[A[n_]] := U_i[] U_i[A[n]]; DU_i[x] := U_i[] U_i[x];
DU_i[y] := U_i[y] U_i[]; DU_i[b] := U_i[b] U_i[]; DU_i[B[n_]] := U_i[B[n]] U_i[];
DU_i[] := U_i[] U_i[];
```

```
DU1 /@ {a, A[1], x, y, b, B[-1]}
{U-1[] U1[a], U-1[] U1[A[1]], U-1[] U1[x], U-1[y] U1[], U-1[b] U1[], U-1[B[-1]] U1[]}
```


$DU_1[x] DU_2[y] // dm_{1,2 \rightarrow 1} // DUForm$

$$- \frac{DU_1[A[1]]}{\hbar} + \frac{DU_1[B[1]]}{\hbar} + DU_1[y, x]$$

$z = U_{-1}[] U_1[x] U_{-2}[y] U_2[] U_{-3}[b] U_3[];$

$(z // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (z // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1}) // DUForm$

0

$\$TD = 5;$

$z = U_{-1}[y, b, b] U_1[A[2], a, x, x] U_{-2}[B[-1], y, y, b, b] U_2[a] U_{-3}[y, y, b] U_3[a, a, x, x, x];$

$(z // dm_{1,2 \rightarrow 1} // dm_{1,3 \rightarrow 1}) - (z // dm_{2,3 \rightarrow 2} // dm_{1,2 \rightarrow 1}) // DUForm$

0

$(\# \rightarrow DUForm[dS_1[DU_1[\#]]) \& /@ \{a, x, y, b\}$

$\{a \rightarrow -DU_1[a], x \rightarrow -e^{-\alpha \beta \hbar} DU_1[A[-1], x], y \rightarrow -DU_1[y, B[-1]], b \rightarrow -DU_1[b]\}$

$(DU_1[x] DU_2[y] // dm_{1,2 \rightarrow 1} // dS_1) - (DU_1[x] DU_2[y] // dS_1 // dS_2 // dm_{2,1 \rightarrow 1}) // ExpandAB$

0

Coassociativity

$(lhs = DU_1[\#] // d\Delta_{1 \rightarrow 1, 2} // d\Delta_{2 \rightarrow 2, 3}; rhs = DU_1[\#] // d\Delta_{1 \rightarrow 1, 3} // d\Delta_{1 \rightarrow 1, 2}; lhs == rhs) \& /@$
 $\{a, x, y, b\}$

{True, True, True, True}

dΔ is algebra morphism

$z = U_{-1}[y, y, y, b, B[2]] U_1[x] U_{-2}[y, b, b, B[-3]] U_2[a, x];$

$(z // dm_{1,2 \rightarrow 1} // d\Delta_{1 \rightarrow 1, 2}) - (z // d\Delta_{2 \rightarrow 3, 4} // d\Delta_{1 \rightarrow 1, 2} // dm_{1,3 \rightarrow 1} // dm_{2,4 \rightarrow 2})$

0

dS is algebra anti-morphism

dS is the convolution inverse of dm.

$z = U_{-1}[y, y, y, b, B[-3]] U_1[a, x, x];$

$\{z // d\Delta_{1 \rightarrow 1, 2} // dS_2 // dm_{1,2 \rightarrow 1}, z // d\Delta_{1 \rightarrow 1, 2} // dS_1 // dm_{1,2 \rightarrow 1}\}$

{0, 0}

The R-Matrix

Quesne's formulas:

$$q = e^{\hbar \alpha \beta}; e_{q_-}[x_-] := \text{Exp}\left[\sum_{k=1}^{\$TD} \frac{(1-q)^k}{k(1-q^k)} x^k\right];$$

Table[Together@SeriesCoefficient[e_ρ[x], {x, 0, n}], {n, 0, \$TD}]

$$\left\{1, 1, \frac{1}{1+\rho}, \frac{1}{(1+\rho)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)}, \frac{1}{(1+\rho)^2(1+\rho^2)(1+\rho+\rho^2)(1+\rho+\rho^2+\rho^3+\rho^4)}\right\}$$

```
R_{i_,j_} := O[U_{-i}[y → y, b → b] U_i[] U_{-j}[] U_j[a → a, x → x], e^{ħ b a} e_q[ħ x y]]
```

\$TD = 3; R_{1,2} // DUForm

$$\begin{aligned} & DU_1[] DU_2[] + \hbar DU_1[b] DU_2[a] + \hbar DU_1[y] DU_2[x] + \frac{1}{2} \hbar^2 DU_1[b, b] DU_2[a, a] + \hbar^2 DU_1[y, b] DU_2[a, x] + \\ & \frac{1}{2} \hbar^2 DU_1[y, y] DU_2[x, x] - \frac{1}{4} \alpha \beta \hbar^3 DU_1[y, y] DU_2[x, x] + \frac{1}{6} \hbar^3 DU_1[b, b, b] DU_2[a, a, a] + \\ & \frac{1}{2} \hbar^3 DU_1[y, b, b] DU_2[a, a, x] + \frac{1}{2} \hbar^3 DU_1[y, y, b] DU_2[a, x, x] + \frac{1}{6} \hbar^3 DU_1[y, y, y] DU_2[x, x, x] \end{aligned}$$

\$TD = 5; {lhs = R_{1,3} // dΔ_{1→1,2}, rhs = R_{2,3} ** R_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm

0

\$TD = 5; {lhs = R_{1,2} // dΔ_{2→2,3}, rhs = R_{1,2} ** R_{1,3}, lhs - rhs} // Last // ExpandAB // DUForm

0

```
$TD = 5; Table[
  {lhs = R_{1,2} ** dΔ_{1→2,1}[f], rhs = dΔ_{1→1,2}[f] ** R_{1,2}, lhs - rhs} // Last // ExpandAB // DUForm,
  {f, {U_{-1}[] U_1[a], U_{-1}[] U_1[x], U_{-1}[b] U_1[], U_{-1}[y] U_1[]}}]
{0, 0, 0, 0}
```

\$TD = 5;

```
{lhs = ExpandAB[R_{1,2} ** R_{1,3} ** R_{2,3}],
  rhs = ExpandAB[R_{2,3} ** R_{1,3} ** R_{1,2}], Coefficient[lhs - rhs, ħ^{TD}]} // Last
```

0

\$TD = 5; Si_2[R_{1,2}] ** R_{1,2} // ExpandAB // DUForm

DU_1[] DU_2[]

Cuaps (Unfinished)

Drinfeld element *u*.

```
u_{i_} := R_{1,2} // dS_1 // dm_{2,1→i}
```

`$TD = 2; u1 // ExpandAB // DUForm`

$$\begin{aligned} & DU_1[] - \beta \hbar DU_1[a] + \alpha \hbar DU_1[b] + \frac{1}{2} \beta^2 \hbar^2 DU_1[a, a] - \hbar DU_1[b, a] - \alpha \beta \hbar^2 DU_1[b, a] + \\ & \frac{1}{2} \alpha^2 \hbar^2 DU_1[b, b] - \hbar DU_1[y, x] + \alpha \beta \hbar^2 DU_1[y, x] + \beta \hbar^2 DU_1[b, a, a] - \alpha \hbar^2 DU_1[b, b, a] - \\ & 2 \alpha \hbar^2 DU_1[y, b, x] + \frac{1}{2} \hbar^2 DU_1[b, b, a, a] + \hbar^2 DU_1[y, b, a, x] + \frac{1}{2} \hbar^2 DU_1[y, y, x, x] \end{aligned}$$

Conjugation by the Drinfeld element implements the square of the antipode.

`$TD = 3; Table[DU1[f] ** u1 == u1 ** dS1[dS1[DU1[f]]] // ExpandAB, {f, {a, x, b, y}}]`
`{True, True, True, True}`

Inverse of the Drinfeld element

$$u_{i-} := R_{1,2} // dS_2 // dS_2 // dm_{2,1 \rightarrow i}$$

`$TD = 3; ui1 ** u1 // ExpandAB`

$$U_{-1}[] U_1[]$$

Drinfeld commutes with its antipode and the product is central

`$TD = 3; u1 ** dS1[u1] - dS1[u1] ** u1 // ExpandAB`

$$0$$

`$TD = 3;`

`(u1 ** dS1[u1] ** DU1[#] - DU1[#] ** u1 ** dS1[u1]) & /@ {a, x, y, b} // ExpandAB`

$$\{0, 0, 0, 0\}$$

Multiplying the inverse and antipode of Drinfeld

`$TD = 3; ui1 ** dS1[u1] - U_{-1}[B[1]] U_1[A[-1]] // ExpandAB // DUForm`

$$0$$

This implies that $dS[u] = uBA^{-1}$. In other words, $u dS[u] =$

$u^2 BA^{-1}$ so we can take the square root and call it v . This is the ribbon element.

$$v_{i-} := \left(0[U_{-i}[b \rightarrow b] U_i[a \rightarrow a], \text{Exp}\left[\frac{\hbar}{2}(-\alpha b + \beta a)\right] \right) ** u_i$$

`$TD = 3; u1 ** dS1[u1] - v1 ** v1 // ExpandAB`

$$0$$

v is supposed to be the Reidemeister 1 curl (with the appropriate cuaps/spinners added!). Note how it too is central.

`$TD = 3; (v1 ** DU1[#] - DU1[#] ** v1) & /@ {a, x, y, b} // ExpandAB`

$$\{0, 0, 0, 0\}$$

The rule for the cuaps is to add to a right-moving cup uv^{-1} and to a right – moving cap vu^{-1} . Note how

$$v u^{-1} = B^{1/2} A^{-1/2}$$

$$\$TD = 3; v_1 ** u_{i_1} - \mathcal{O} [U_{-1}[b \to b] U_1[a \to a], \text{Exp}[\frac{\hbar}{2} (-\alpha b + \beta a)]] // \text{ExpandAB}$$

0

$$\text{cap}_{i_-} := \mathcal{O} [U_{-i}[b \to b] U_i[a \to a], \text{Exp}[\frac{\hbar}{2} (-\alpha b + \beta a)]]$$

$$\text{cup}_{i_-} := \mathcal{O} [U_{-i}[b \to b] U_i[a \to a], \text{Exp}[-\frac{\hbar}{2} (-\alpha b + \beta a)]]$$

Now let's do R2 and oppositely oriented Reidemeister 2:

$$\$TD = 3; dS_2[R_{1,2}] R_{3,4} // dm_{1,3 \to 1} // dm_{2,4 \to 2} // \text{ExpandAB} // \text{DUForm}$$

DU₁ [] DU₂ []

$$\$TD = 3;$$

$$dS_2[R_{1,2}] R_{3,4} \text{cup}_5 \text{cap}_6 // dm_{1,3 \to 1} // dm_{4,5 \to 4} // dm_{4,2 \to 4} // dm_{4,6 \to 4} // \text{ExpandAB} // \text{DUForm}$$

DU₁ [] DU₄ []

Rotate a crossing:

$$\$TD = 3;$$

$$(\text{cup}_1 dS_5[R_{2,5}] \text{cap}_3 \text{cup}_4 \text{cap}_6 // dm_{1,2 \to 1} // dm_{1,3 \to 1} // dm_{4,5 \to 4} // dm_{4,6 \to 4}) - dS_4[R_{1,4}] // \text{ExpandAB}$$

0

Reidemeister 1 curls, the two positives agree with the two negatives and they are inverses and agree with v, v^{-1} .

$$\$TD = 2; \{$$

$$\begin{aligned} & (R_{1,2} \text{cap}_3 // dm_{1,3 \to 1} // dm_{1,2 \to 1}) - (R_{1,2} \text{cup}_3 // dm_{2,3 \to 2} // dm_{2,1 \to 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \to 1} // dm_{1,2 \to 1}) - (dS_2[R_{1,2}] \text{cap}_3 // dm_{2,3 \to 2} // dm_{2,1 \to 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \to 1} // dm_{1,2 \to 1}) ** (R_{1,2} \text{cap}_3 // dm_{1,3 \to 1} // dm_{1,2 \to 1}), \\ & (dS_2[R_{1,2}] \text{cup}_3 // dm_{1,3 \to 1} // dm_{1,2 \to 1}) - v_1 \} // \text{ExpandAB} // \text{DUForm} \end{aligned}$$

{0, 0, DU₁ [], 0}

The Central Element

$$c_1 = \alpha U_{-1}[b] U_1[] + \beta U_{-1}[] U_1[a];$$

$$UB[c_1, \#] \& /@ \{U_{-1}[] U_1[a], U_{-1}[] U_1[x], U_{-1}[y] U_1[], U_{-1}[b] U_1[]\}$$

{0, 0, 0, 0}

Commuting Exponentials

Commuting e^a with e^x :

$$\$TD = 5; \mathcal{O} [U_1[x \to x, a \to a], e^{\hbar (\mu x + \nu a)}] == \mathcal{O} [U_1[a \to a, x \to x], e^{\hbar (\mu e^{-\hbar \alpha \nu} x + \nu a)}]$$

True

Commuting e^b with e^y :

$$\text{\$TD} = 5; \text{\textcircled{O}} [U_1[b \rightarrow b, y \rightarrow y], e^{\hbar(\mu y + \nu b)}] == \text{\textcircled{O}} [U_1[y \rightarrow y, b \rightarrow b], e^{\hbar(\mu e^{-\hbar\beta\nu} y + \nu b)}]$$

True

The co-product of e^a :

$$\text{\$TD} = 5; \Delta_{1 \rightarrow 1, 2} [\text{\textcircled{O}} [U_1[a \rightarrow a], e^{\hbar\mu a}]] == \text{\textcircled{O}} [U_1[a_1 \rightarrow a] U_2[a_2 \rightarrow a], e^{\hbar\mu a_1} e^{\hbar\mu a_2}]$$

True

The co-product of e_q^x :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \Delta_{1 \rightarrow 1, 2} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[a_1 \rightarrow a, x_1 \rightarrow x] U_2[x_2 \rightarrow x], e_q[\hbar\mu e^{-\hbar\beta a_1} x_2] e_q[\hbar\mu x_1]]$$

True

The triple co-product of e_q^x :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \Delta_{1 \rightarrow 1, 2, 3} // \text{ExpandAB}) - \text{\textcircled{O}} [U_1[a_1 \rightarrow a, x_1 \rightarrow x] U_2[a_2 \rightarrow a, x_2 \rightarrow x] U_3[x_3 \rightarrow x], e_q[\hbar\mu e^{-\hbar\beta(a_1+a_2)} x_3] e_q[\hbar\mu e^{-\hbar\beta a_1} x_2] e_q[\hbar\mu x_1]]$$

0

The co-product of e_q^y :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[y \rightarrow y], e_q[\hbar\nu y]] // \Delta_{1 \rightarrow 1, 2} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[y_1 \rightarrow y] U_2[y_2 \rightarrow y, b_2 \rightarrow b], e_q[\hbar\nu y_2] e_q[\hbar\nu e^{-\hbar\alpha b_2} y_1]]$$

True

The triple co-product of e_q^y :

$$\text{\$TD} = 5; (\text{\textcircled{O}} [U_1[y \rightarrow y], e_q[\hbar\nu y]] // \Delta_{1 \rightarrow 1, 2, 3} // \text{ExpandAB}) == \text{\textcircled{O}} [U_1[y_1 \rightarrow y] U_2[y_2 \rightarrow y, b_2 \rightarrow b] U_3[y_3 \rightarrow y, b_3 \rightarrow b], e_q[\hbar\nu y_3] e_q[\hbar\nu e^{-\hbar\alpha b_3} y_2] e_q[\hbar\nu e^{-\hbar\alpha(b_2+b_3)} y_1]]$$

True

The inverse-antipode on e_q^x :

$$\text{\$TD} = 8; (\text{\textcircled{O}} [U_1[x \rightarrow x], e_q[\hbar\mu x]] // \text{Si}_1 // \text{ExpandAB}) - \left(\text{\textcircled{O}} [U_1[a \rightarrow a, x \rightarrow x], \text{Sum} \left[\frac{(-\hbar\mu)^k q^{-k(k+1)/2} e^{\hbar\beta k a} x^k}{\text{QFactorial}[k, q]}, \{k, 0, \text{\$TD}\} \right] \right)$$

0

Pairing e_q^x with e_q^y :

```

$TD = 5;
$TD *= 2;
lhs = O[U_{-1}[y → y], e_q[ħ v y]] O[U_1[x → x], e_q[ħ μ x]] // Expand // P_{1,-1} // ExpandAB;
($TD /= 2; e_q[ħ μ v] - lhs // ExpandAB)
0
    
```

Computing $e^{\mu x} y e^{-\mu x}$.

Following BBS:VanDerVeen-170620-212242, though with explicit \hbar put into μ, v .

```

phi_0[x_] := e^x;
phi_k[x_] := Module[{t}, Simplify[ $\frac{e^t - \text{Normal@Series}[e^t, \{t, 0, k - 1\}]}{t^k}$  /. t → x]];
    
```

```
Table[phi_k[x], {k, 0, 3}]
```

$$\left\{ e^x, \frac{-1 + e^x}{x}, \frac{-1 + e^x - x}{x^2}, -\frac{2 - 2e^x + 2x + x^2}{2x^3} \right\}$$

\$TD = 5;

GS = T μ (A⁻¹ φ₁ [ħ μ (q - 1) x] - 1) - μ (A φ₁ [ħ μ (q⁻¹ - 1) x] - 1) /. {A -> e^{-ħβa}};

(G = O[U₋₁[] U₁[x -> x, a -> a], GS]) // DUForm

{lhs = O[U₋₁[] U₁[x -> x], e^{ħμx}] ** DU₁[y] ** O[U₋₁[] U₁[x -> x], e^{-ħμx}],

rhs = Expand[DU₁[y] + μ (T - 1) DU₁[] + G],

lhs - rhs /. T * ε -> DU₁[A[1]] ** DU₁[B[1]] ** ε} // Last // ExpandAB // DUForm

$$\begin{aligned} & \beta \mu \hbar \text{DU}_1[a] + T \beta \mu \hbar \text{DU}_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] + \frac{1}{12} \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] + \frac{1}{12} T \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] + \\ & \frac{1}{48} \alpha^4 \beta^4 \mu^2 \hbar^5 \text{DU}_1[x] - \frac{1}{48} T \alpha^4 \beta^4 \mu^2 \hbar^5 \text{DU}_1[x] - \frac{1}{2} \beta^2 \mu \hbar^2 \text{DU}_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 \text{DU}_1[a, a] - \\ & \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] - \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \\ & \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \frac{1}{12} \alpha^3 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, x] + \frac{1}{12} T \alpha^3 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, x] - \\ & \frac{1}{6} \alpha^2 \beta^2 \mu^3 \hbar^4 \text{DU}_1[x, x] + \frac{1}{6} T \alpha^2 \beta^2 \mu^3 \hbar^4 \text{DU}_1[x, x] - \frac{1}{6} \alpha^3 \beta^3 \mu^3 \hbar^5 \text{DU}_1[x, x] - \\ & \frac{1}{6} T \alpha^3 \beta^3 \mu^3 \hbar^5 \text{DU}_1[x, x] + \frac{1}{6} \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \frac{1}{6} T \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \\ & \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{8} \alpha^2 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, x] - \\ & \frac{1}{8} T \alpha^2 \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, x] + \frac{1}{6} \alpha^2 \beta^3 \mu^3 \hbar^5 \text{DU}_1[a, x, x] + \frac{1}{6} T \alpha^2 \beta^3 \mu^3 \hbar^5 \text{DU}_1[a, x, x] - \\ & \frac{1}{24} \beta^4 \mu \hbar^4 \text{DU}_1[a, a, a, a] + \frac{1}{24} T \beta^4 \mu \hbar^4 \text{DU}_1[a, a, a, a] - \frac{1}{12} \alpha \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, a, x] + \\ & \frac{1}{12} T \alpha \beta^4 \mu^2 \hbar^5 \text{DU}_1[a, a, a, x] + \frac{1}{120} \beta^5 \mu \hbar^5 \text{DU}_1[a, a, a, a, a] + \frac{1}{120} T \beta^5 \mu \hbar^5 \text{DU}_1[a, a, a, a, a] \end{aligned}$$

0

K = 3; \$TD = 5;

GS1 = $\mu T (A^{-1} \phi_1 [\hbar \mu (q - 1) x] - 1) - \mu (A \phi_1 [\hbar \mu (q^{-1} - 1) x] - 1) /. \{A \rightarrow e^{-\hbar \beta a}\};$

(G1 = $\mathcal{O}[U_{-1}[] U_1[x \rightarrow x, a \rightarrow a], GS1]$) // DUForm

Collect[ExpandAB[

Expand[$DU_1[y] + (T - 1) \mu DU_1[] + G1 - \mathcal{O}[U_{-1}[] U_1[x \rightarrow x], e^{\hbar \mu x}] ** DU_1[y] **$

$\mathcal{O}[U_{-1}[] U_1[x \rightarrow x], e^{-\hbar \mu x}] /. T * \mathcal{E}_- \rightarrow \mathcal{E} ** DU_1[A[1]] ** DU_1[B[1]]$

],

$\beta]$

$$\begin{aligned} & \beta \mu \hbar DU_1[a] + T \beta \mu \hbar DU_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 DU_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 DU_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 DU_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 DU_1[x] + \frac{1}{12} \alpha^3 \beta^3 \mu^2 \hbar^4 DU_1[x] + \frac{1}{12} T \alpha^3 \beta^3 \mu^2 \hbar^4 DU_1[x] + \\ & \frac{1}{48} \alpha^4 \beta^4 \mu^2 \hbar^5 DU_1[x] - \frac{1}{48} T \alpha^4 \beta^4 \mu^2 \hbar^5 DU_1[x] - \frac{1}{2} \beta^2 \mu \hbar^2 DU_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 DU_1[a, a] - \\ & \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 DU_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 DU_1[a, x] - \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 DU_1[a, x] - \\ & \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 DU_1[a, x] - \frac{1}{12} \alpha^3 \beta^4 \mu^2 \hbar^5 DU_1[a, x] + \frac{1}{12} T \alpha^3 \beta^4 \mu^2 \hbar^5 DU_1[a, x] - \\ & \frac{1}{6} \alpha^2 \beta^2 \mu^3 \hbar^4 DU_1[x, x] + \frac{1}{6} T \alpha^2 \beta^2 \mu^3 \hbar^4 DU_1[x, x] - \frac{1}{6} \alpha^3 \beta^3 \mu^3 \hbar^5 DU_1[x, x] - \\ & \frac{1}{6} T \alpha^3 \beta^3 \mu^3 \hbar^5 DU_1[x, x] + \frac{1}{6} \beta^3 \mu \hbar^3 DU_1[a, a, a] + \frac{1}{6} T \beta^3 \mu \hbar^3 DU_1[a, a, a] + \\ & \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 DU_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 DU_1[a, a, x] + \frac{1}{8} \alpha^2 \beta^4 \mu^2 \hbar^5 DU_1[a, a, x] - \\ & \frac{1}{8} T \alpha^2 \beta^4 \mu^2 \hbar^5 DU_1[a, a, x] + \frac{1}{6} \alpha^2 \beta^3 \mu^3 \hbar^5 DU_1[a, x, x] + \frac{1}{6} T \alpha^2 \beta^3 \mu^3 \hbar^5 DU_1[a, x, x] - \\ & \frac{1}{24} \beta^4 \mu \hbar^4 DU_1[a, a, a, a] + \frac{1}{24} T \beta^4 \mu \hbar^4 DU_1[a, a, a, a] - \frac{1}{12} \alpha \beta^4 \mu^2 \hbar^5 DU_1[a, a, a, x] + \\ & \frac{1}{12} T \alpha \beta^4 \mu^2 \hbar^5 DU_1[a, a, a, x] + \frac{1}{120} \beta^5 \mu \hbar^5 DU_1[a, a, a, a, a] + \frac{1}{120} T \beta^5 \mu \hbar^5 DU_1[a, a, a, a, a] \end{aligned}$$

0

K = 3; \$TD = 5;

GS2 = Collect[Normal@Series[GS1, {β, 0, K}], β, Factor];

(G2 = 0[U₋₁[] U₁[x → x], a → a], GS2)] // DUForm

Collect[ExpandAB[

Expand[DU₁[y] + (T - 1) μ DU₁[] + G2 - 0[U₋₁[] U₁[x → x], e^{ħ μ x}] ** DU₁[y] **

0[U₋₁[] U₁[x → x], e^{-ħ μ x}]] /. T * ε₋ => ε ** DU₁[A[1]] ** DU₁[B[1]]

],

β]

$$\begin{aligned} & \beta \mu \hbar \text{DU}_1[a] + T \beta \mu \hbar \text{DU}_1[a] + \frac{1}{2} \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \frac{1}{2} T \alpha \beta \mu^2 \hbar^2 \text{DU}_1[x] + \\ & \frac{1}{4} \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{1}{4} T \alpha^2 \beta^2 \mu^2 \hbar^3 \text{DU}_1[x] - \frac{5}{24} \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] - \frac{5}{24} T \alpha^3 \beta^3 \mu^2 \hbar^4 \text{DU}_1[x] - \\ & \frac{1}{2} \beta^2 \mu \hbar^2 \text{DU}_1[a, a] + \frac{1}{2} T \beta^2 \mu \hbar^2 \text{DU}_1[a, a] - \frac{1}{2} \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] + \frac{1}{2} T \alpha \beta^2 \mu^2 \hbar^3 \text{DU}_1[a, x] - \\ & \frac{1}{4} \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] - \frac{1}{4} T \alpha^2 \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, x] + \frac{1}{6} \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \\ & \frac{1}{6} T \beta^3 \mu \hbar^3 \text{DU}_1[a, a, a] + \frac{1}{4} \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] + \frac{1}{4} T \alpha \beta^3 \mu^2 \hbar^4 \text{DU}_1[a, a, x] \\ & \frac{1}{6} \alpha^3 \beta^2 \mu^3 \hbar^5 U_{-1}[b] U_1[x, x] + \\ & \beta^3 \left(-\frac{7}{12} \alpha^3 \mu^2 \hbar^4 U_{-1}[] U_1[x] + \frac{7}{24} \alpha^4 \mu^2 \hbar^5 U_{-1}[b] U_1[x] - \frac{1}{6} \alpha^2 \mu^3 \hbar^5 U_{-1}[] U_1[a, x, x] \right) + \\ & \beta^4 \left(-\frac{7}{24} \alpha^4 \mu^2 \hbar^5 U_{-1}[] U_1[x] + \frac{7}{24} \alpha^3 \mu^2 \hbar^5 U_{-1}[] U_1[a, x] + \frac{1}{24} \alpha \mu \hbar^5 U_{-1}[b] U_1[a, a, a, a] \right) + \\ & \frac{1}{40} \beta^5 \mu \hbar^5 U_{-1}[] U_1[a, a, a, a, a] \end{aligned}$$

Clear[ħ]

K = 3; \$TD = 5;

GS1 = μ T (A⁻¹ φ₁[ħ μ (q - 1) x] - 1) - μ (A φ₁[ħ μ (q⁻¹ - 1) x] - 1) /. {A → e^{-ħ β a}}

$$-\mu \left(-1 + \frac{e^{-a \beta \hbar} (-1 + e^{(-1 + e^{-a \beta \hbar}) x \mu \hbar})}{(-1 + e^{-\alpha \beta \hbar}) x \mu \hbar} \right) + T \mu \left(-1 + \frac{e^{a \beta \hbar} (-1 + e^{(-1 + e^{\alpha \beta \hbar}) x \mu \hbar})}{(-1 + e^{\alpha \beta \hbar}) x \mu \hbar} \right)$$

K = 3;

GS2 = Collect[Normal@Series[GS1, {β, 0, K}], β, Factor]

$$\begin{aligned} & \frac{1}{2} (1 + T) \beta \mu \hbar (2 a + x \alpha \mu \hbar) + \frac{1}{4} (-1 + T) \beta^2 \mu \hbar^2 (2 a^2 + 2 a x \alpha \mu \hbar + x \alpha^2 \mu \hbar) + \\ & \frac{1}{24} (1 + T) \beta^3 \mu \hbar^3 (4 a^3 + 6 a^2 x \alpha \mu \hbar + 6 a x \alpha^2 \mu \hbar - 5 x \alpha^3 \mu \hbar) \end{aligned}$$

Series[GS1 - GS2, {ħ, 0, \$TD}]

$$\frac{1}{24} \left(-a^4 \beta^4 \mu + a^4 T \beta^4 \mu + 7 x \alpha^3 \beta^3 \mu^2 + 7 T x \alpha^3 \beta^3 \mu^2 - 4 x^2 \alpha^2 \beta^2 \mu^3 + 4 T x^2 \alpha^2 \beta^2 \mu^3 \right) \hbar^4 +$$

$$\frac{1}{240} \left(2 a^5 \beta^5 \mu + 2 a^5 T \beta^5 \mu - 20 a^3 x \alpha \beta^4 \mu^2 + 20 a^3 T x \alpha \beta^4 \mu^2 - 30 a^2 x \alpha^2 \beta^4 \mu^2 + \right.$$

$$\left. 30 a^2 T x \alpha^2 \beta^4 \mu^2 - 20 a x \alpha^3 \beta^4 \mu^2 + 20 a T x \alpha^3 \beta^4 \mu^2 - 5 x \alpha^4 \beta^4 \mu^2 + 5 T x \alpha^4 \beta^4 \mu^2 + \right.$$

$$\left. 40 a x^2 \alpha^2 \beta^3 \mu^3 + 40 a T x^2 \alpha^2 \beta^3 \mu^3 + 40 x^2 \alpha^3 \beta^3 \mu^3 + 40 T x^2 \alpha^3 \beta^3 \mu^3 \right) \hbar^5 + O[\hbar]^6$$

Series[Normal@Series[GS1 - GS2, {ħ, 0, \$TD}], {β, 0, K}]

$$\frac{1}{6} \left(-x^2 \alpha^2 \mu^3 \hbar^4 + T x^2 \alpha^2 \mu^3 \hbar^4 \right) \beta^2 + \frac{1}{24}$$

$$\left(7 x \alpha^3 \mu^2 \hbar^4 + 7 T x \alpha^3 \mu^2 \hbar^4 + 4 a x^2 \alpha^2 \mu^3 \hbar^5 + 4 a T x^2 \alpha^2 \mu^3 \hbar^5 + 4 x^2 \alpha^3 \mu^3 \hbar^5 + 4 T x^2 \alpha^3 \mu^3 \hbar^5 \right) \beta^3 + O[\beta]^4$$

Normal@Series[GS1 - GS2, {ħ, 0, \$TD}] + O[β]^{K+1}

$$\frac{1}{6} \left(-x^2 \alpha^2 \mu^3 \hbar^4 + T x^2 \alpha^2 \mu^3 \hbar^4 \right) \beta^2 + \frac{1}{24}$$

$$\left(7 x \alpha^3 \mu^2 \hbar^4 + 7 T x \alpha^3 \mu^2 \hbar^4 + 4 a x^2 \alpha^2 \mu^3 \hbar^5 + 4 a T x^2 \alpha^2 \mu^3 \hbar^5 + 4 x^2 \alpha^3 \mu^3 \hbar^5 + 4 T x^2 \alpha^3 \mu^3 \hbar^5 \right) \beta^3 + O[\beta]^4$$

Series[GS1 - GS2, {β, 0, K}]

$$O[\beta]^4$$