$$a = -\begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix} \frac{\alpha}{\rho - \sigma};\ b = -\begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix} \frac{\beta}{\rho - \sigma};$$

$$\hbar = -\frac{(\rho - \sigma)}{\alpha\,\beta};$$

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{e^{-\rho} - e^{-\sigma}}{-\hbar};\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

```
(A = MatrixExp[-ℏ β a]; B = MatrixExp[-ℏ α b];)
```

```
{a.x - x.a == -α x, a.y - y.a == α y} // Simplify
```

```
{True, True}
```

```
{b.y - y.b == -β y, x.b - b.x == -β x} // Simplify
```

```
{True, True}
```

```
β a + α b // Simplify // MatrixForm
```

$$\begin{pmatrix} -\frac{\alpha\,\beta\,(\rho+\sigma)}{\rho-\sigma} & 0 \\ 0 & -\frac{\alpha\,\beta\,(\rho+\sigma)}{\rho-\sigma} \end{pmatrix}$$

$$x.y - y.x == \frac{B - A}{\hbar}\ \text{// Simplify}$$

```
True
```

```
MatrixExp[λ1 y].MatrixExp[λ2 (x.y - y.x)].MatrixExp[λ3 x] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} & \frac{e^{-\rho+\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}-\sigma}\,(-e^{\rho}+e^{\sigma})\,\alpha\,\beta\,\lambda 3}{\rho-\sigma} \\ e^{\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}}\,\lambda 1 & e^{\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} + \frac{e^{-\rho+\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}-\sigma}\,(-e^{\rho}+e^{\sigma})\,\alpha\,\beta\,\lambda 1\,\lambda 3}{\rho-\sigma} \end{pmatrix}$$

```
eqn = Simplify[
  MatrixExp[μ x].MatrixExp[ν y] - MatrixExp[λ1 y].MatrixExp[λ2 (x.y - y.x)].MatrixExp[λ3 x]]
```

$$\Big\{\Big\{1 - e^{\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} + \frac{(e^{-\rho} - e^{-\sigma})\,\alpha\,\beta\,\mu\,\nu}{\rho - \sigma},\ \frac{1}{\rho - \sigma} e^{-\rho-\sigma}\,(-e^{\rho} + e^{\sigma})\,\alpha\,\beta\,\Big(-e^{\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}}\,\lambda 3 + \mu\Big)\Big\},$$
$$\Big\{-e^{\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}}\,\lambda 1 + \nu,\ 1 - e^{\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} + \frac{1}{\rho - \sigma} e^{-\rho+\frac{(e^{-\rho}-e^{-\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}-\sigma}\,(e^{\rho} - e^{\sigma})\,\alpha\,\beta\,\lambda 1\,\lambda 3\Big\}\Big\}$$

```
eqns = Flatten[eqn]
```

$$\Big\{1 - e^{-\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} + \frac{e^{-\rho-\sigma}\,(-e^{\rho} + e^{\sigma})\,\alpha\,\beta\,\mu\,\nu}{\rho - \sigma},$$
$$-\frac{1}{\rho - \sigma} e^{-\rho-\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}-\sigma}\,(-e^{\rho} + e^{\sigma})\,\alpha\,\beta\,\lambda 3 + \frac{e^{-\rho-\sigma}\,(-e^{\rho} + e^{\sigma})\,\alpha\,\beta\,\mu}{\rho - \sigma},$$
$$-e^{-\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}}\,\lambda 1 + \nu,\ 1 - e^{\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}} - \frac{1}{\rho - \sigma} e^{-\rho-\frac{e^{-\rho-\sigma}\,(e^{\rho}-e^{\sigma})\,\alpha\,\beta\,\lambda 2}{\rho-\sigma}-\sigma}\,(-e^{\rho} + e^{\sigma})\,\alpha\,\beta\,\lambda 1\,\lambda 3\Big\}$$

```
Series[eqns, {β, 0, 1}]
```

$$\left\{\frac{e^{-\rho-\sigma}\left(e^{\rho}-e^{\sigma}\right)\alpha\left(\lambda 2-\mu\nu\right)\beta}{\rho-\sigma}+O[\beta]^2,\ \frac{e^{-\rho-\sigma}\left(e^{\rho}-e^{\sigma}\right)\alpha\left(\lambda 3-\mu\right)\beta}{\rho-\sigma}+O[\beta]^2,\ \left(-\lambda 1+\nu\right)+\frac{e^{-\rho-\sigma}\left(e^{\rho}-e^{\sigma}\right)\alpha\,\lambda 1\,\lambda 2\,\beta}{\rho-\sigma}+O[\beta]^2,\ \frac{e^{-\rho-\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\left(\lambda 2-\lambda 1\,\lambda 3\right)\beta}{\rho-\sigma}+O[\beta]^2\right\}$$

```
Series[eqns /. {λ1 → ν + c β, λ2 → μ ν, λ3 → μ}, {β, 0, 1}]
```

$$\left\{O[\beta]^2,\ O[\beta]^2,\ \left(-c-\frac{e^{-\rho}\alpha\mu\nu^2}{\rho-\sigma}+\frac{e^{-\sigma}\alpha\mu\nu^2}{\rho-\sigma}\right)\beta+O[\beta]^2,\ O[\beta]^2\right\}$$

```
c /. Solve[-c - (e^-ρ α μ ν^2)/(ρ - σ) + (e^-σ α μ ν^2)/(ρ - σ) == 0, c] // Simplify
```

$$\left\{\frac{e^{-\rho-\sigma}\left(e^{\rho}-e^{\sigma}\right)\alpha\mu\nu^2}{\rho-\sigma}\right\}$$

```
ser = Series[
  eqns /. {λ1 → ν + (e^(-ρ-σ) (e^ρ - e^σ) α μ ν^2)/(ρ - σ) β + c1 β^2, λ2 → μ ν + c2 β, λ3 → μ + c3 β}, {β, 0, 2}]
```

$$\left\{\frac{1}{2}\left(-\frac{e^{-2\rho-2\sigma}\left(e^{\rho}-e^{\sigma}\right)^2\alpha^2\mu^2\nu^2}{(\rho-\sigma)^2}+\frac{2\,c2\,e^{-\rho-\sigma}\left(e^{\rho}-e^{\sigma}\right)\alpha}{\rho-\sigma}\right)\beta^2+O[\beta]^3,\right.$$

$$\frac{1}{\rho-\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\left(-c3\,e^{-\rho-\sigma}-\frac{e^{-2\rho-2\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\mu^2\nu}{\rho-\sigma}\right)\beta^2+O[\beta]^3,$$

$$\left(-c1+\frac{e^{-2\rho}\alpha^2\mu^2\nu^3}{2\,(\rho-\sigma)^2}-\frac{e^{-\rho-\sigma}\alpha^2\mu^2\nu^3}{(\rho-\sigma)^2}+\frac{e^{-2\sigma}\alpha^2\mu^2\nu^3}{2\,(\rho-\sigma)^2}-\frac{c2\,e^{-\rho}\alpha\nu}{\rho-\sigma}+\frac{c2\,e^{-\sigma}\alpha\nu}{\rho-\sigma}\right)\beta^2+O[\beta]^3,$$

$$\left.-\left(\left(\left(e^{-2\rho-2\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\left(-e^{\rho}\alpha\mu^2\nu^2+e^{\sigma}\alpha\mu^2\nu^2-2\,c2\,e^{\rho+\sigma}\rho+2\,c3\,e^{\rho+\sigma}\nu\rho+2\,c2\,e^{\rho+\sigma}\sigma-2\,c3\,e^{\rho+\sigma}\nu\sigma\right)\right)\beta^2\right)\Big/\left(2\,(\rho-\sigma)^2\right)\right)+O[\beta]^3\right\}$$

```
Solve[SeriesCoefficient[#, 2] == 0 & /@ ser, {c1, c2, c3}]
```

$$\left\{\left\{c1\to\frac{e^{-2\rho-2\sigma}\left(e^{\rho}-e^{\sigma}\right)^2\alpha^2\mu^2\nu^3}{(\rho-\sigma)^2},\ c2\to-\frac{e^{-\rho-\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\mu^2\nu^2}{2\,(\rho-\sigma)},\ c3\to-\frac{e^{-\rho-\sigma}\left(-e^{\rho}+e^{\sigma}\right)\alpha\mu^2\nu}{\rho-\sigma}\right\}\right\}$$