Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-04/.

Issues:

1. S does not invert R. (Perhaps because H must be interpreted as $e^{\gamma h}$).
2. dm is not meta-associative.
3. R doesn't satisfy YB.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-05"];
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\eta g}$ and to $H = e^{\gamma h}$, where $[g, e] = \gamma\, e$ and $[h, f] = -\eta\, f$.
Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2,\ G_1\, G_2,\ e_1 + G_1\, e_2,\ h_1 + h_2,\ H_1\, H_2,\ f_1\, H_2 + f_2)$.
Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;
$PBWRule = {G[_] → 1, g → 2, e → 3, H[_] → 4, h → 5, f → 6};
```

```
B[g, e] = γ e; B[e, G[n_]] = (e^(-n γ η) - 1) U[G[n], e]; B[g, G[_]] = 0;
B[h, f] = -η f; B[f, H[n_]] = (e^(n γ η) - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on penseive://Projects/UEA/.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≼ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[] := U_i[]; UU_i_[1] := U_i[];
UU_i_[x_[n_]^p_] := U_i[x[n p]];
UU_i_[x_^p_] := UU_i @@ Table[x, {p}];
UU_i_[ℰ_] := ℰ /. {
    U[xs__] :⇢ U_i[xs],
    x : AlgebraAtom :⇢ U_i[x]
   };
UU_i_[x_, xs__] := UU_t1[x] UU_t2[xs] // Expand // m_{t1,t2→i};
USimp[sd_SeriesData] := MapAt[USimp, sd, {3, All}];
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
m_s__[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_[n1_]] U_j_[x_[n2_], yy___]] :=
  USimp[c If[n1 + n2 == 0, U_i[xx] U_j[yy], U_i[xx, x[n1 + n2]] U_j[yy]] // m_i,j→k];
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j→i) U_j[yy] // Expand // m_i,j→k)
     c // USimp
  ];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :> i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[
   {Sx = Supp[x], Sy = Supp[y], is, σ, z},
   If[MatchQ[Sx ⋃ Sy, {_Integer ...}] && Min[Sx ⋃ Sy] < 0,
    is = Abs[Sx] ⋂ Abs[Sy];
    z = x; Do[z = m_-i→-σ@i[m_i→σ@i[z]], {i, is}];
    z = USimp[y z]; Do[z = dm_σ@i,i→i[z], {i, is}];
    z,
    (* else *) is = Sx ⋂ Sy;
    z = x; Do[z = m_i→σ@i[z], {i, is}];
    z = USimp[y z]; Do[z = m_σ@i,i→i[z], {i, is}];
    z
   ]
  ];
UB[x_, y_] := USimp[x ** y - y ** x];
```

```
𝕆[specs_, sd_SeriesData] := MapAt[𝕆[specs, #] &, sd, {3, All}];
𝕆[specs_, poly_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] :> Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[poly, vars] /. (ps_ → c_) :> c (
        specs /. MapThread[((#1 → _) :> #3^#2) &, {vars, ps, elems}] /. U_i_ :> UU_i
       )]
 ]
```

The 2D Lie BiAlgebra Pencil, Testing

```
𝔒[U₁[x → g], Normal@Series[e^(η x), {η, 0, 5}]]
```

$$U_1[\,] + \eta\, U_1[g] + \frac{1}{2}\eta^2\, U_1[g, g] + \frac{1}{6}\eta^3\, U_1[g, g, g] + \frac{1}{24}\eta^4\, U_1[g, g, g, g] + \frac{1}{120}\eta^5\, U_1[g, g, g, g, g]$$

```
With[{G = 𝔒[U₁[x → g], Series[e^(η x), {η, 0, 5}]]}, UB[U₁[e], G] - (e^(-γ η) - 1) G ** U₁[e]]
```

$$O[\eta]^6$$

```
B[e, G[3]]
```

$$\left(-1 + e^{-3\gamma\eta}\right) U[G[3], e]$$

```
With[{H = 𝔒[U₁[x → h], Series[e^(γ x), {γ, 0, 5}]]}, UB[U₁[f], H] - (e^(γ η) - 1) H ** U₁[f]]
```

$$O[\gamma]^6$$

```
x = U₁[g, G[2], e, e, e] U₂[g, g, e] U₃[g, g, G[-3], e];
(x // m₁,₂→₁ // m₁,₃→₁) - (x // m₂,₃→₂ // m₁,₂→₁)
```

$$0$$

```
x = U₁[h, H[2], f, f, f] U₂[h, h, f] U₃[h, h, H[-3], f];
(x // m₁,₂→₁ // m₁,₃→₁) - (x // m₂,₃→₂ // m₁,₂→₁)
```

$$0$$

The Co-Product and Co-Associativity

```
qΔ_i_→j_,k_[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
      U_i[] → U_j[] U_k[],
      U_i[g, xs___] :>
       (USimp[(U_j[g] U_k[] + U_j[] U_k[g]) qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k),
      U_i[G[n_], xs___] :> (USimp[U_j[G[n]] U_k[G[n]] qΔ_i→tj,tk[U_i[xs]]] //
          m_j,tj→j // m_k,tk→k),
      U_i[e, xs___] :> (USimp[(U_j[e] U_k[G[1]] + U_j[] U_k[e]) qΔ_i→tj,tk[U_i[xs]]] //
          m_j,tj→j // m_k,tk→k),
      U_i[h, xs___] :> (USimp[(U_j[h] U_k[] + U_j[] U_k[h]) qΔ_i→tj,tk[U_i[xs]]] //
          m_j,tj→j // m_k,tk→k),
      U_i[H[n_], xs___] :> (USimp[U_j[H[n]] U_k[H[n]] qΔ_i→tj,tk[U_i[xs]]] //
          m_j,tj→j // m_k,tk→k),
      U_i[f, xs___] :> (USimp[(U_j[f] U_k[] + U_j[H[1]] U_k[f]) qΔ_i→tj,tk[U_i[xs]]] //
          m_j,tj→j // m_k,tk→k)
     }];
```

```
qΔ_i_→j_,k_,l__[ℰ_] := ℰ // qΔ_i→j,k // qΔ_k→k,l
```

```
U₁[e] // qΔ₁→₁,₂
```

```
U₁[ ] U₂[e] + U₁[e] U₂[G[1]]
```

```
{lhs = U₁[e] // qΔ₁→₁,₂ // qΔ₂→₂,₃, rhs = U₁[e] // qΔ₁→₁,₃ // qΔ₁→₁,₂, lhs == rhs}
```

```
{U₁[ ] U₂[ ] U₃[e] + U₁[ ] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]],
 U₁[ ] U₂[ ] U₃[e] + U₁[ ] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]], True}
```

```
U₁[f] // qΔ₁→₁,₂
```

```
U₁[f] U₂[ ] + U₁[H[1]] U₂[f]
```

```
{lhs = U₁[f] // qΔ₁→₁,₂ // qΔ₂→₂,₃, rhs = U₁[f] // qΔ₁→₁,₃ // qΔ₁→₁,₂, lhs == rhs}
```

```
{U₁[f] U₂[ ] U₃[ ] + U₁[H[1]] U₂[f] U₃[ ] + U₁[H[1]] U₂[H[1]] U₃[f],
 U₁[f] U₂[ ] U₃[ ] + U₁[H[1]] U₂[f] U₃[ ] + U₁[H[1]] U₂[H[1]] U₃[f], True}
```

```
x = U₁[g, G[2], e, e, e] U₂[g, g, G[-3], e];
(x // m₁,₂→₁ // qΔ₁→₁,₂) - (x // qΔ₂→₃,₄ // qΔ₁→₁,₂ // m₁,₃→₁ // m₂,₄→₂)
```

```
0
```

```
x = U₁[h, H[2], f, f, f] U₂[h, h, H[-3], f];
(x // m₁,₂→₁ // qΔ₁→₁,₂) - (x // qΔ₂→₃,₄ // qΔ₁→₁,₂ // m₁,₃→₁ // m₂,₄→₂)
```

```
0
```

The Antipode

Why o why this annoyance of left-vs-right?

```
S[g] = -g; S[G[n_]] := G[-n]; S[e] = -𝔢^(𝛾 𝜂) U[G[-1], e];
S[h] = -h; S[H[n_]] := H[-n]; S[f] = -U[H[-1], f];
Sᵢ_[ℰ_] := Module[{ti}, USimp[
    ℰ /. Uᵢ[x_, xs___] :> mₜᵢ,ᵢ→ᵢ[Expand[UUᵢ[S[x]] Sₜᵢ[Uₜᵢ[xs]]]]
  ]];
```

```
{lhs = S₁[U₁[e]], rhs = -U₁[e] ** U₁[G[-1]], lhs == rhs}
```

```
{-𝔢^(𝛾 𝜂) U₁[G[-1], e], -𝔢^(𝛾 𝜂) U₁[G[-1], e], True}
```

```
U₁[e] // S₁ // S₁
```

```
𝔢^(𝛾 𝜂) U₁[e]
```

```
U₁[f] // S₁ // S₁
```

```
𝔢^(𝛾 𝜂) U₁[f]
```

```
S₁[U₁[g, G[3], e, e]]
```

```
2 𝔢^(9 𝛾 𝜂) 𝛾 U₁[G[-5], e, e] - 𝔢^(9 𝛾 𝜂) U₁[G[-5], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2
```

```
U1[G[3], g, e, e] U2[G[5]] + U1[G[3], g, e] U2[G[4], e] +
 ℯ^(-γ η) U1[G[3], g, e] U2[G[4], e] + U1[G[3], e, e] U2[G[5], g] + U1[G[3], g] U2[G[3], e, e] +
 U1[G[3], e] U2[G[4], g, e] + ℯ^(-γ η) U1[G[3], e] U2[G[4], g, e] + U1[G[3]] U2[G[3], g, e, e]
```

```
U1[g, G[3], e, e] // qΔ1→1,2 // S2
```

```
U1[G[3], g, e, e] U2[G[-5]] - ℯ^(4 γ η) γ U1[G[3], e] U2[G[-5], e] - ℯ^(5 γ η) γ U1[G[3], e] U2[G[-5], e] -
 ℯ^(4 γ η) U1[G[3], g, e] U2[G[-5], e] - ℯ^(5 γ η) U1[G[3], g, e] U2[G[-5], e] -
 U1[G[3], e, e] U2[G[-5], g] + 2 ℯ^(9 γ η) γ U1[G[3]] U2[G[-5], e, e] +
 ℯ^(9 γ η) U1[G[3], g] U2[G[-5], e, e] + ℯ^(4 γ η) U1[G[3], e] U2[G[-5], g, e] +
 ℯ^(5 γ η) U1[G[3], e] U2[G[-5], g, e] - ℯ^(9 γ η) U1[G[3]] U2[G[-5], g, e, e]
```

```
test = U1[g, G[3], e, e];
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
 test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, 2 ℯ^(6 γ η) γ U1[e, e] - 2 ℯ^(7 γ η) γ U1[e, e] - 2 ℯ^(8 γ η) γ U1[e, e] + 2 ℯ^(9 γ η) γ U1[e, e]}
```

```
test = U1[h, H[3], f, f];
{test // qΔ1→1,2 // S2 // m1,2→1, test // qΔ1→1,2 // S2 // m2,1→1,
 test // qΔ1→1,2 // S1 // m1,2→1, test // qΔ1→1,2 // S1 // m2,1→1}
```

```
{0, 0, 0, -2 ℯ^(-10 γ η) η U1[H[-2], f, f] +
  2 ℯ^(-9 γ η) η U1[H[-2], f, f] + 2 ℯ^(-8 γ η) η U1[H[-2], f, f] - 2 ℯ^(-7 γ η) η U1[H[-2], f, f]}
```

```
x = U1[h, H[2], f, f, f] U2[h, h, H[-3], f];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
```

```
0
```

```
x = U1[g, G[2], e, e, e] U2[g, g, G[-3], e];
(x // m1,2→1 // S1) - (x // S1 // S2 // m2,1→1)
```

```
0
```

```
x = U1[];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

```
U1[]
```

```
x = U1[];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

```
U1[]
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

```
0
```

```
x = U1[g, G[2], e, e, e];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

```
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S1 // m1,2→1)
```

```
0
```

```
x = U1[h, H[2], f, f, f];
(x // qΔ1→1,2 // S2 // m1,2→1)
```

```
0
```

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[]] = 1;
P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
{{P[U[g], U[h]] = 1, P[U[g], U[H[n_]]] = n γ, P[U[g], U[f]] = 0},
 {P[U[G[n_]], U[h]] = n η, P[U[G[n_]], U[H[m_]]] = e^(n m η γ), P[U[G[_]], U[f]] = 0},
 {P[U[e], U[h]] = 0, P[U[e], U[H[_]]] = 0, P[U[e], U[f]] = 1}};
```

```
Pi_,j_[ℰ_] := USimp[ℰ /. Ui[xs___] Uj[ys___] → P[U[xs], U[ys]]];
```

```
t = Ui[g] Uj[e] Uk[f];
{mi,j→i[t] - mj,i→i[t], qΔk→k,1[t] - qΔk→1,k[t]}
```

```
{γ Ui[e] Uk[f], Ui[g] Uj[e] Uk[f] U1[] - Ui[g] Uj[e] Uk[] U1[f] +
  Ui[g] Uj[e] Uk[H[1]] U1[f] - Ui[g] Uj[e] Uk[f] U1[H[1]]}
```

```
t = Ui[g] Uj[e] Uk[f];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1}
```

```
{γ, γ}
```

```
Table[t = Ui[xi] Uj[xj] Uk[yk];
 {(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1},
 {xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {γ, γ}}}, {{{0, 0}, {-γ, -γ}}, {{0, 0}, {0, 0}}}}
```

```
Table[t = Ui[xi] Uk[yk] U1[yl];
 {(qΔi→i,j[t] - qΔi→j,i[t]) // Pi,k // Pj,1, (mk,1→k[t] - m1,k→k[t]) // Pi,k},
 {xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}, {{{0, 0}, {-η, -η}}, {{η, η}, {0, 0}}}}
```

General Pairings

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[x_, xs__], U[y_]] := P[U[x, xs], U[y]] =
   Module[{i, j, k, l}, USimp[U_i[x] U_j[xs] qΔ_{k→k,l}[U_k[y]]] // P_{i,k} // P_{j,l}];
P[U[xs__], U[y_, ys__]] := P[U[xs], U[y, ys]] =
   Module[{i, j, k, l}, USimp[qΔ_{i→i,j}[U_i[xs]] U_k[y] U_l[ys]] // P_{i,k} // P_{j,l}];
```

```
{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
```

$\{0, \gamma, 0\}$

```
P[U[e], U[f, f]]
```

0

```
P[U[e, e], U[f, f]]
```

$1 + e^{\gamma \eta}$

```
lhs = Factor@Table[P[U @@ Table[e, {n}], U @@ Table[f, {n}]], {n, 7}]
```

$$\left\{1,\ 1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\right.$$
$$\left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right),$$
$$\left(1+e^{\gamma\eta}\right)^3\left(1+e^{2\gamma\eta}\right)\left(1-e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)^2\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right),$$
$$\left(1+e^{\gamma\eta}\right)^3\left(1+e^{2\gamma\eta}\right)\left(1-e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)^2$$
$$\left.\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}+e^{5\gamma\eta}+e^{6\gamma\eta}\right)\right\}$$

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^{γ η}], {n, 7}]
```

$$\left\{1,\ 1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\right.$$
$$\left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right),$$
$$\left(1+e^{\gamma\eta}\right)^3\left(1+e^{2\gamma\eta}\right)\left(1-e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)^2\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right),$$
$$\left(1+e^{\gamma\eta}\right)^3\left(1+e^{2\gamma\eta}\right)\left(1-e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)^2$$
$$\left.\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}+e^{5\gamma\eta}+e^{6\gamma\eta}\right)\right\}$$

```
lhs == rhs
```

True

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

120

```
P[U[g, g, g, g, g, e, e, e, e], U[h, h, h, h, h, f, f, f, f]] // Factor
```

$120\left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)$

```
x = U_1[g, G[-3], e, e] U_2[g, g, G[1], e] U_3[h, H[2], f, f, f];
(x // m_{1,2→1} // P_{1,3}) - (x // qΔ_{3→3,4} // P_{1,3} // P_{2,4})
```

0

```
x = U_1[h, H[-3], f, f] U_2[h, h, H[1], f] U_3[g, G[2], e, e, e];
(x // m_{1,2→1} // P_{3,1}) - (x // qΔ_{3→3,4} // P_{3,1} // P_{4,2})
```

0

```
x = U1[g, g, G[-3], e, e, e] U2[h, h, H[2], f, f, f];
(x // S1 // P1,2) - (x // S2 // P1,2)
```

```
0
```

The Double

```
dmi_,j_→k_[ℰ_] := Module[{t1, t2, t3, h1, h2, h3},
  ℰ // qΔi→h1,h2,h3 // Sh1 // qΔ-j→t1,t2,t3 // Ph1,t1 // Ph3,t3 // mh2,j→k // m-i,t2→-k]
```

```
U-1[] U1[g] U-2[h] U2[] // dm1,2→1
```

```
U-1[h] U1[g]
```

```
U-1[] U1[g] U-2[f] U2[] // dm1,2→1
```

```
-γ U-1[f] U1[] + U-1[f] U1[g]
```

```
U-1[] U1[G[1]] U-2[f] U2[] // dm1,2→1
```

```
ℯ^(-γ η) U-1[f] U1[G[1]]
```

```
U-1[] U1[e] U-2[h] U2[] // dm1,2→1
```

```
η U-1[] U1[e] + U-1[h] U1[e]
```

```
U-1[] U1[e] U-2[H[1]] U2[] // dm1,2→1
```

```
ℯ^(γ η) U-1[H[1]] U1[e]
```

```
3□ U-1[] U1[e] U-2[f] U2[] // dm1,2→1
```

```
3□ U-1[H[1]] U1[] + 3□ U-1[f] U1[e] - 3□ U-1[] U1[G[1]]
```

```
x = U-1[] U1[e] U-2[f] U2[] U-3[h] U3[];
(x // dm1,2→1 // dm1,3→1) - (x // dm2,3→2 // dm1,2→1)
```

```
-η U-1[H[1]] U1[] + η U-1[] U1[G[1]]
```

```
x = U-1[h, h, f] U1[G[2], e, e] U-2[H[-1], f, f] U2[g] U-3[h, f] U3[g, g, e];
(x // dm1,2→1 // dm1,3→1) - (x // dm2,3→2 // dm1,2→1)
```

```
2 ℯ^(-5 γ η) γ^3 η U-1[h, h, f, f, f] U1[G[2], e, e] - 4 ℯ^(-4 γ η) γ^3 η U-1[h, h, f, f, f] U1[G[2], e, e] +
 2 ℯ^(-3 γ η) γ^3 η U-1[h, h, f, f, f] U1[G[2], e, e] + 8 ℯ^(-2 γ η) γ^3 η U-1[h, h, f, f, f] U1[G[2], e, e] -
 2 ℯ^(-4 γ η) γ^3 U-1[h, h, h, f, f, f] U1[G[2], e, e] + 2 ℯ^(-2 γ η) γ^3 U-1[h, h, h, f, f, f] U1[G[2], e, e] -
 2 ℯ^(-9 γ η) γ^3 η U-1[H[-1], h, h, f, f, f] U1[G[3], e, e] +
 4 ℯ^(-8 γ η) γ^3 η U-1[H[-1], h, h, f, f, f] U1[G[3], e, e] -
 2 ℯ^(-7 γ η) γ^3 η U-1[H[-1], h, h, f, f, f] U1[G[3], e, e] -
 8 ℯ^(-6 γ η) γ^3 η U-1[H[-1], h, h, f, f, f] U1[G[3], e, e] +
 2 ℯ^(-8 γ η) γ^3 U-1[H[-1], h, h, h, f, f, f] U1[G[3], e, e] -
 2 ℯ^(-6 γ η) γ^3 U-1[H[-1], h, h, h, f, f, f] U1[G[3], e, e] -
 5 ℯ^(-5 γ η) γ^2 η U-1[h, h, f, f, f] U1[G[2], g, e, e] +
 10 ℯ^(-4 γ η) γ^2 η U-1[h, h, f, f, f] U1[G[2], g, e, e] -
 5 ℯ^(-3 γ η) γ^2 η U-1[h, h, f, f, f] U1[G[2], g, e, e] -
 20 ℯ^(-2 γ η) γ^2 η U-1[h, h, f, f, f] U1[G[2], g, e, e] +
 5 ℯ^(-4 γ η) γ^2 U-1[h, h, h, f, f, f] U1[G[2], g, e, e] -
```

$$
\begin{aligned}
&5\,e^{-2\gamma\eta}\,\gamma^2\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,e,e]+\\
&\gamma\,\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,e]+4\,e^{\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,e]+\\
&3\,e^{2\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,e]-\\
&\gamma\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,e]-e^{-\gamma\eta}\,\gamma\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,e]+\\
&e^{\gamma\eta}\,\gamma\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,e]+\\
&e^{2\gamma\eta}\,\gamma\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,e]+\\
&5\,e^{-9\gamma\eta}\,\gamma^2\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]-\\
&10\,e^{-8\gamma\eta}\,\gamma^2\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]+\\
&5\,e^{-7\gamma\eta}\,\gamma^2\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]+\\
&20\,e^{-6\gamma\eta}\,\gamma^2\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,e,e]-\\
&5\,e^{-8\gamma\eta}\,\gamma^2\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,e,e]+\\
&5\,e^{-6\gamma\eta}\,\gamma^2\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,e,e]-3\,\gamma\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]-\\
&e^{-3\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]-5\,e^{-2\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]-\\
&7\,e^{-\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,e]-\gamma\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]+\\
&e^{-4\gamma\eta}\,\gamma\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]+2\,e^{-3\gamma\eta}\,\gamma\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]-\\
&2\,e^{-\gamma\eta}\,\gamma\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,e]+e^{-5\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]+\\
&4\,e^{-4\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]+\\
&3\,e^{-3\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,e]-\\
&e^{-6\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]-\\
&e^{-5\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\\
&e^{-4\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\\
&e^{-3\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,e]+\\
&4\,e^{-5\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\\
&8\,e^{-4\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\\
&4\,e^{-3\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\\
&16\,e^{-2\gamma\eta}\,\gamma\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\\
&4\,e^{-4\gamma\eta}\,\gamma\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,e,e]+\\
&4\,e^{-2\gamma\eta}\,\gamma\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,e,e]-\\
&\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,g,e]-4\,e^{\gamma\eta}\,\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,g,e]-\\
&3\,e^{2\gamma\eta}\,\eta\,U_{-1}[H[1],h,h,f,f]\,U_1[G[2],g,g,g,e]+\\
&U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,g,e]+\\
&e^{-\gamma\eta}\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,g,e]-\\
&e^{\gamma\eta}\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,g,e]-\\
&e^{2\gamma\eta}\,U_{-1}[H[1],h,h,h,f,f]\,U_1[G[2],g,g,g,e]-\\
&4\,e^{-9\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,e,e]+\\
&8\,e^{-8\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,e,e]-\\
&4\,e^{-7\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,e,e]-\\
&16\,e^{-6\gamma\eta}\,\gamma\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,e,e]+\\
&4\,e^{-8\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,g,e,e]-\\
&4\,e^{-6\gamma\eta}\,\gamma\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,g,e,e]+\\
&3\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,g,e]+e^{-3\gamma\eta}\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,g,e]+\\
&5\,e^{-2\gamma\eta}\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,g,e]+7\,e^{-\gamma\eta}\,\eta\,U_{-1}[h,h,f,f]\,U_1[G[3],g,g,g,e]+\\
&U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,g,e]-e^{-4\gamma\eta}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,g,e]-\\
&2\,e^{-3\gamma\eta}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,g,e]+2\,e^{-\gamma\eta}\,U_{-1}[h,h,h,f,f]\,U_1[G[3],g,g,g,e]-\\
&e^{-5\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,g,e]-\\
&4\,e^{-4\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,g,e]-\\
&3\,e^{-3\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f]\,U_1[G[4],g,g,g,e]+\\
&e^{-6\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,g,e]+\\
&e^{-5\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,g,e]-\\
&e^{-4\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,g,e]-\\
&e^{-3\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f]\,U_1[G[4],g,g,g,e]-
\end{aligned}
$$

$$
\begin{aligned}
&e^{-5\gamma\eta}\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] +\\
&2\,e^{-4\gamma\eta}\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] -\\
&e^{-3\gamma\eta}\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] -\\
&4\,e^{-2\gamma\eta}\,\eta\,U_{-1}[h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] +\\
&e^{-4\gamma\eta}\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] -\\
&e^{-2\gamma\eta}\,U_{-1}[h,h,h,f,f,f]\,U_1[G[2],g,g,g,e,e] +\\
&e^{-9\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,g,e,e] -\\
&2\,e^{-8\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,g,e,e] +\\
&e^{-7\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,g,e,e] +\\
&4\,e^{-6\gamma\eta}\,\eta\,U_{-1}[H[-1],h,h,f,f,f]\,U_1[G[3],g,g,g,e,e] -\\
&e^{-8\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,g,g,e,e] +\\
&e^{-6\gamma\eta}\,U_{-1}[H[-1],h,h,h,f,f,f]\,U_1[G[3],g,g,g,e,e]
\end{aligned}
$$

The R-Matrix

Using Quesne's formula.

```
R_{i_,j_}[d_] := Module[{x, y}, ℴ[
   U_{-i}[x_1 → h, x_2 → f] U_i[] U_{-j}[] U_j[y_1 → g, y_2 → e],
   Series[Exp[ħ x_1 y_1 + Sum_{k=1}^{d} (1 - e^{ħ η γ})^k (ħ x_2 y_2)^k / (k (1 - e^{k ħ η γ}))], {ħ, 0, d}]
  ]]
```

$$R_{1,2}[1]$$

$$U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[] + \left(U_{-2}[]\,U_{-1}[f]\,U_1[]\,U_2[e] + U_{-2}[]\,U_{-1}[h]\,U_1[]\,U_2[g]\right)\hbar + O[\hbar]^2$$

$$R_{1,2}[1] \;//\; S_{-1}$$

$$U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[] + \left(-U_{-2}[]\,U_{-1}[H[-1],f]\,U_1[]\,U_2[e] - U_{-2}[]\,U_{-1}[h]\,U_1[]\,U_2[g]\right)\hbar + O[\hbar]^2$$

$$R_{1,2}[1]\,R_{3,4}[1] \;//\; S_{-3}$$

$$\begin{aligned}
&U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[]\,U_3[]\,U_4[] +\\
&\Big(U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[f]\,U_1[]\,U_2[e]\,U_3[]\,U_4[] + U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[h]\,U_1[]\,U_2[g]\,U_3[]\,U_4[] -\\
&\quad U_{-4}[]\,U_{-3}[H[-1],f]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[]\,U_3[]\,U_4[e] -\\
&\quad U_{-4}[]\,U_{-3}[h]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[]\,U_3[]\,U_4[g]\Big)\hbar + O[\hbar]^2
\end{aligned}$$

$$R_{1,2}[1]\,R_{3,4}[1] \;//\; S_4 \;//\; m_{1,3\to1} \;//\; m_{2,4\to2}$$

$$\begin{aligned}
&U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[] +\\
&\Big(U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[f]\,U_1[]\,U_2[e] - U_{-4}[]\,U_{-3}[h]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[g] +\\
&\quad U_{-4}[]\,U_{-3}[]\,U_{-2}[]\,U_{-1}[h]\,U_1[]\,U_2[g] - e^{\gamma\eta}\,U_{-4}[]\,U_{-3}[f]\,U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[G[-1],e]\Big)\hbar + O[\hbar]^2
\end{aligned}$$

$$S_{-1}[R_{1,2}[1]] \;**\; R_{1,2}[1]$$

$$U_{-2}[]\,U_{-1}[]\,U_1[]\,U_2[] + \left(U_{-2}[]\,U_{-1}[f]\,U_1[]\,U_2[e] - U_{-2}[]\,U_{-1}[H[-1],f]\,U_1[]\,U_2[e]\right)\hbar + O[\hbar]^2$$

```
R_{1,2}[1] ** R_{1,3}[1] ** R_{2,3}[1]
```

```
U_{-3}[] U_{-2}[] U_{-1}[] U_1[] U_2[] U_3[] +
  (U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[e] U_3[] + U_{-3}[] U_{-2}[] U_{-1}[h] U_1[] U_2[g] U_3[] +
     U_{-3}[] U_{-2}[f] U_{-1}[] U_1[] U_2[] U_3[e] + U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[] U_3[e] +
     U_{-3}[] U_{-2}[h] U_{-1}[] U_1[] U_2[] U_3[g] + U_{-3}[] U_{-2}[] U_{-1}[h] U_1[] U_2[] U_3[g]) ħ + O[ħ]^2
```

```
R_{2,3}[1] ** R_{1,3}[1] ** R_{1,2}[1]
```

```
U_{-3}[] U_{-2}[] U_{-1}[] U_1[] U_2[] U_3[] +
  (U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[e] U_3[] + U_{-3}[] U_{-2}[] U_{-1}[h] U_1[] U_2[g] U_3[] +
     U_{-3}[] U_{-2}[f] U_{-1}[] U_1[] U_2[] U_3[e] + U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[] U_3[e] +
     U_{-3}[] U_{-2}[h] U_{-1}[] U_1[] U_2[] U_3[g] + U_{-3}[] U_{-2}[] U_{-1}[h] U_1[] U_2[] U_3[g]) ħ + O[ħ]^2
```

```
With[{d = 1}, R_{1,2}[d] ** R_{1,3}[d] ** R_{2,3}[d] == R_{2,3}[d] ** R_{1,3}[d] ** R_{1,2}[d]]
```

```
True
```

```
With[{d = 2}, USimp[R_{1,2}[d] ** R_{1,3}[d] ** R_{2,3}[d] - R_{2,3}[d] ** R_{1,3}[d] ** R_{1,2}[d]]]
```

```
(-γ U_{-3}[] U_{-2}[h] U_{-1}[f] U_1[] U_2[] U_3[e] +
     U_{-3}[] U_{-2}[H[1]] U_{-1}[f] U_1[] U_2[] U_3[e] - η U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[g] U_3[e] -
     U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[G[1]] U_3[e] + 2 η U_{-3}[] U_{-2}[] U_{-1}[f] U_1[] U_2[e] U_3[g]) ħ^2 + O[ħ]^3
```