

Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-04/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-05"];
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{ng}$ and to $H = e^{hf}$, where $[g, e] = \gamma e$ and $[h, f] = -\eta f$.

Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2, G_1 G_2, e_1 + G_1 e_2, h_1 + h_2, H_1 H_2, f_1 H_2 + f_2)$.

Also, (g, e) and (h, f) are dual bases.

```
AlgebraAtom = g | G[_] | e | h | H[_] | f;
$PBWRule = {G[_] -> 1, g -> 2, e -> 3, H[_] -> 4, h -> 5, f -> 6};
```

```
B[g, e] = \gamma e; B[e, G[n_]] = (e^{-n \gamma \eta} - 1) U[G[n], e]; B[g, G[_]] = 0;
B[h, f] = -\eta f; B[f, H[n_]] = (e^{n \eta \gamma} - 1) U[H[n], f]; B[h, H[_]] = 0;
```

UEA with provisional modification

This section is based on pensieve://Projects/UEA/.

```
B[0, _] = 0; B[_ , 0] = 0;
B[c_ * x : AlgebraAtom, y_] := Expand[c B[x, y]];
B[y_, c_ * x : AlgebraAtom] := Expand[c B[y, x]];
B[x_Plus, y_] := B[# , y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ <= y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[1] := U_i[];
UU_i_[x_[n_]^{p_}] := U_i[x[n p]];
UU_i_[x^{p_}] := UU_i@@Table[x, {p}];
UU_i_[\mathcal{E}_] := \mathcal{E} /. {
  U[xs_] => U_i[xs],
  x : AlgebraAtom => U_i[x]
};
UU_i_[x_, xs_] := UU_{t1}[x] UU_{t2}[xs] // Expand // m_{t1, t2 -> i};
USimp[\mathcal{E}_] := Collect[\mathcal{E}, Times[U[___] ..], Expand];
USimp[\mathcal{E}_] := Expand[\mathcal{E}];
```

```
m_s_[0] = 0;
m_s_[x_Plus] := m_s_ /@ x;
m_s_[sd_SeriesData] := MapAt[m_s, sd, {3, All}];
m_{i->j}[\mathcal{E}_] := \mathcal{E} /. U_i -> U_j;
```

```

mi,j→k[c_. Ui[x___] Uj[ ]] := c Uk[x];
mi,j→k[c_. Ui[ ] Uj[y___]] := c Uk[y];
mi,j→k[c_. Ui[xx___, x_[n1_]] Uj[x_[n2_], yy___]] :=
  USimp[c If[n1 + n2 == 0, Ui[xx] Uj[yy], Ui[xx, x[n1 + n2]] Uj[yy]] // mi,j→k];
mi,j→k[c_. Ui[xx___, x_] Uj[y_, yy___]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((Ui[xx] (Uj[y, x] + UUj[B[x, y]])) // Expand // mi,j→i) Uj[yy] // Expand // mi,j→k)
  c // USimp
];

```

```

Supp[ε_] := Union@Cases[{ε}, Ui[___] := i, ∞];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ∩ Supp[y], σ, z},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = mσ@i,i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];

```

```

O[specs_, sd_SeriesData] := MapAt[O[specs, #] &, sd, {3, All}];
O[specs_, poly_] := Module[{rules, vars, elems},
  rules = Union@@Cases[{specs}, U[u___] := Cases[{u}, r_Rule], ∞];
  vars = First/@rules; elems = Last/@rules;
  USimp@Total[CoefficientRules[poly, vars] /. (ps_ → c_) := c (
    specs /. MapThread[({#1 → _} := #3#2) &, {vars, ps, elems}] /. Ui := UUi
  )]
];

```

The 2D Lie BiAlgebra Pencil, Testing

```

O[U1[x → g], Normal@Series[eηx, {η, 0, 5}]]
U1[ ] + η U1[g] +  $\frac{1}{2} \eta^2 U_1[g, g]$  +  $\frac{1}{6} \eta^3 U_1[g, g, g]$  +  $\frac{1}{24} \eta^4 U_1[g, g, g, g]$  +  $\frac{1}{120} \eta^5 U_1[g, g, g, g, g]$ 

With[{G = O[U1[x → g], Series[eηx, {η, 0, 5}]]}, UB[U1[e], G] - (e-γ η - 1) G ** U1[e]]
O[η]6

B[e, G[3]]
(-1 + e-3γ η) U[G[3], e]

With[{H = O[U1[x → h], Series[eγx, {γ, 0, 5}]]}, UB[U1[f], H] - (eγ η - 1) H ** U1[f]]
O[γ]6

```

The Co-Product and Co-Associativity

```

qΔi→j,k[ε-] := USimp@Module[{tj, tk}, ε /. {
  Ui[] → Uj[] Uk[],
  Ui[g, xS____] =>
    (USimp[(Uj[g] Uk[] + Uj[] Uk[g]) qΔi→tj,tk[Ui[xS]]] // mj,tj→j // mk,tk→k),
  Ui[G[n-], xS____] => (USimp[Uj[G[n]] Uk[G[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[e, xS____] => (USimp[(Uj[e] Uk[G[1]] + Uj[] Uk[e]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[h, xS____] => (USimp[(Uj[h] Uk[] + Uj[] Uk[h]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[H[n-], xS____] => (USimp[Uj[H[n]] Uk[H[n]] qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k),
  Ui[f, xS____] => (USimp[(Uj[f] Uk[] + Uj[H[1]] Uk[f]) qΔi→tj,tk[Ui[xS]]] //
    mj,tj→j // mk,tk→k)
}];

```

```

qΔi→j,k,l[ε-] := ε // qΔi→j,k // qΔk→l

```

U₁[e] // qΔ_{1→1,2}

U₁[] U₂[e] + U₁[e] U₂[G[1]]

{lhs = U₁[e] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U₁[e] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}

{U₁[] U₂[] U₃[e] + U₁[] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]],
 U₁[] U₂[] U₃[e] + U₁[] U₂[e] U₃[G[1]] + U₁[e] U₂[G[1]] U₃[G[1]], True}

U₁[f] // qΔ_{1→1,2}

U₁[f] U₂[] + U₁[H[1]] U₂[f]

{lhs = U₁[f] // qΔ_{1→1,2} // qΔ_{2→2,3}, rhs = U₁[f] // qΔ_{1→1,3} // qΔ_{1→1,2}, lhs == rhs}

{U₁[f] U₂[] U₃[] + U₁[H[1]] U₂[f] U₃[] + U₁[H[1]] U₂[H[1]] U₃[f],
 U₁[f] U₂[] U₃[] + U₁[H[1]] U₂[f] U₃[] + U₁[H[1]] U₂[H[1]] U₃[f], True}

The Antipode

Why o why this annoyance of left-vs-right?

```

S[g] = -g; S[G[n-]] := G[-n]; S[e] = -eχη U[G[-1], e];
S[h] = -h; S[H[n-]] := H[-n]; S[f] = -U[H[-1], f];
Si[ε-] := Module[{ti}, USimp[
  ε /. Ui[x-, xS____] => mti,i→i[Expand[UUi[S[x]] Sti[Uti[xS]]]]
]];

```

```
{lhs = S1[U1[e]], rhs = -U1[e] ** U1[G[-1]], lhs == rhs}
{-e^{\gamma\eta} U1[G[-1], e], -e^{\gamma\eta} U1[G[-1], e], True}
```

```
U1[e] // S1 // S1
e^{\gamma\eta} U1[e]
```

```
U1[f] // S1 // S1
e^{\gamma\eta} U1[f]
```

```
S1[U1[g, G[3], e, e]]
2 e^{9\gamma\eta} \gamma U1[G[-5], e, e] - e^{9\gamma\eta} U1[G[-5], g, e, e]
```

```
U1[g, G[3], e, e] // q\Delta_{1\to 1,2}
U1[G[3], g, e, e] U2[G[5]] + U1[G[3], g, e] U2[G[4], e] +
e^{-\gamma\eta} U1[G[3], g, e] U2[G[4], e] + U1[G[3], e, e] U2[G[5], g] + U1[G[3], g] U2[G[3], e, e] +
U1[G[3], e] U2[G[4], g, e] + e^{-\gamma\eta} U1[G[3], e] U2[G[4], g, e] + U1[G[3]] U2[G[3], g, e, e]
```

```
U1[g, G[3], e, e] // q\Delta_{1\to 1,2} // S2
U1[G[3], g, e, e] U2[G[-5]] - e^{4\gamma\eta} \gamma U1[G[3], e] U2[G[-5], e] - e^{5\gamma\eta} \gamma U1[G[3], e] U2[G[-5], e] -
e^{4\gamma\eta} U1[G[3], g, e] U2[G[-5], e] - e^{5\gamma\eta} U1[G[3], g, e] U2[G[-5], e] -
U1[G[3], e, e] U2[G[-5], g] + 2 e^{9\gamma\eta} \gamma U1[G[3]] U2[G[-5], e, e] +
e^{9\gamma\eta} U1[G[3], g] U2[G[-5], e, e] + e^{4\gamma\eta} U1[G[3], e] U2[G[-5], g, e] +
e^{5\gamma\eta} U1[G[3], e] U2[G[-5], g, e] - e^{9\gamma\eta} U1[G[3]] U2[G[-5], g, e, e]
```

```
test = U1[g, G[3], e, e];
{test // q\Delta_{1\to 1,2} // S2 // m_{1,2\to 1}, test // q\Delta_{1\to 1,2} // S2 // m_{2,1\to 1},
test // q\Delta_{1\to 1,2} // S1 // m_{1,2\to 1}, test // q\Delta_{1\to 1,2} // S1 // m_{2,1\to 1}}
{0, 0, 0, 2 e^{6\gamma\eta} \gamma U1[e, e] - 2 e^{7\gamma\eta} \gamma U1[e, e] - 2 e^{8\gamma\eta} \gamma U1[e, e] + 2 e^{9\gamma\eta} \gamma U1[e, e]}
```

```
test = U1[h, H[3], f, f];
{test // q\Delta_{1\to 1,2} // S2 // m_{1,2\to 1}, test // q\Delta_{1\to 1,2} // S2 // m_{2,1\to 1},
test // q\Delta_{1\to 1,2} // S1 // m_{1,2\to 1}, test // q\Delta_{1\to 1,2} // S1 // m_{2,1\to 1}}
{0, 0, 0, -2 e^{-10\gamma\eta} \eta U1[H[-2], f, f] +
2 e^{-9\gamma\eta} \eta U1[H[-2], f, f] + 2 e^{-8\gamma\eta} \eta U1[H[-2], f, f] - 2 e^{-7\gamma\eta} \eta U1[H[-2], f, f]}
```

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[]] = 1;
P[U[], U[H[_]]] = P[U[G[_]], U[]] = 1;
P[U[], U[[_]]] = P[U[[_]], U[]] = 0;
(
P[U[g], U[h]] = 1          P[U[g], U[H[n_]]] = n \gamma          P[U[g], U[f]] = 0
P[U[G[n_]], U[h]] = n \eta  P[U[G[n_]], U[H[m_]]] = e^{n m \eta \gamma}  P[U[G[_]], U[f]] = 0
P[U[e], U[h]] = 0          P[U[e], U[H[_]]] = 0                  P[U[e], U[f]] = 1
);
```

```
P_{i,j}[_] := USimp[\epsilon /. U_i[xs_...] U_j[ys_...] \to P[U[xs], U[ys]]];
```

```

t = Ui[g] Uj[e] Uk[f];
{mi,j→i[t] - mj,i→i[t], qΔk→k,1[t] - qΔk→1,k[t]}
{γ Ui[e] Uk[f], Ui[g] Uj[e] Uk[f] U1[ ] - Ui[g] Uj[e] Uk[ ] U1[f] +
  Ui[g] Uj[e] Uk[H[1]] U1[f] - Ui[g] Uj[e] Uk[f] U1[H[1]]}

```

```

t = Ui[g] Uj[e] Uk[f];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1}
{γ, γ}

```

```

Table[t = Ui[xi] Uj[xj] Uk[yk];
{(mi,j→i[t] - mj,i→i[t]) // Pi,k, (qΔk→k,1[t] - qΔk→1,k[t]) // Pi,k // Pj,1},
{xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {γ, γ}}, {{{0, 0}, {-γ, -γ}}, {{0, 0}, {0, 0}}}}

```

```

Table[t = Ui[xi] Uk[yk] U1[yl];
{(qΔi→i,j[t] - qΔi→j,i[t]) // Pi,k // Pj,1, (mk,1→k[t] - m1,k→k[t]) // Pi,k},
{xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{{0, 0}, {-η, -η}}, {{η, η}, {0, 0}}}}

```

General Pairings

The pairing sequence: (one,one) (above), (many,one), (many,many).

```

P[U[x_, xs_], U[y_]] := P[U[x, xs], U[y]] =
  Module[{i, j, k, l}, USimp[Ui[x] Uj[xs] qΔk→k,1[Uk[y]]] // Pi,k // Pj,1];
P[U[xs_], U[y_, ys_]] := P[U[xs], U[y, ys]] =
  Module[{i, j, k, l}, USimp[qΔi→i,j[Ui[xs]] Uk[y] U1[ys]] // Pi,k // Pj,1];

```

```

{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
{0, γ, 0}

```

```
P[U[e], U[f, f]]
```

```
0
```

```
P[U[e, e], U[f, f]]
```

```
1 + eγ η
```

```
lhs = Factor@Table[P[U@@Table[e, {n}], U@@Table[f, {n}]], {n, 7}]
```

```

{1, 1 + eγ η, (1 + eγ η) (1 + eγ η + e2 γ η), (1 + eγ η)2 (1 + e2 γ η) (1 + eγ η + e2 γ η),
(1 + eγ η)2 (1 + e2 γ η) (1 + eγ η + e2 γ η) (1 + eγ η + e2 γ η + e3 γ η + e4 γ η),
(1 + eγ η)3 (1 + e2 γ η) (1 - eγ η + e2 γ η) (1 + eγ η + e2 γ η)2 (1 + eγ η + e2 γ η + e3 γ η + e4 γ η),
(1 + eγ η)3 (1 + e2 γ η) (1 - eγ η + e2 γ η) (1 + eγ η + e2 γ η)2
(1 + eγ η + e2 γ η + e3 γ η + e4 γ η) (1 + eγ η + e2 γ η + e3 γ η + e4 γ η + e5 γ η + e6 γ η)}

```

```
rhs = Simplify@FunctionExpand@Table[QFactorial[n, e^yeta], {n, 7}]
{1, 1 + e^yeta, (1 + e^yeta) (1 + e^yeta + e^2yeta), (1 + e^yeta)^2 (1 + e^2yeta) (1 + e^yeta + e^2yeta),
(1 + e^yeta)^2 (1 + e^2yeta) (1 + e^yeta + e^2yeta) (1 + e^yeta + e^2yeta + e^3yeta + e^4yeta),
(1 + e^yeta)^3 (1 + e^2yeta) (1 - e^yeta + e^2yeta) (1 + e^yeta + e^2yeta)^2 (1 + e^yeta + e^2yeta + e^3yeta + e^4yeta),
(1 + e^yeta)^3 (1 + e^2yeta) (1 - e^yeta + e^2yeta) (1 + e^yeta + e^2yeta)^2
(1 + e^yeta + e^2yeta + e^3yeta + e^4yeta) (1 + e^yeta + e^2yeta + e^3yeta + e^4yeta + e^5yeta + e^6yeta)}
```

lhs == rhs

True

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

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The Double

```
dm_{i,j->k}_{E} := Module[{t1, t2, t3, h1, h2, h3},
  E // qDelta_{i->h1,h2,h3} // S_{h1} // qDelta_{-j->t1,t2,t3} // P_{h1,t1} // P_{h3,t3} // m_{h2,j->k} // m_{-i,t2->-k}
```

```
U_{-1}[] U_1[g] U_{-2}[h] U_2[] // dm_{1,2->1}
```

```
U_{-1}[h] U_1[g]
```

```
U_{-1}[] U_1[g] U_{-2}[f] U_2[] // dm_{1,2->1}
```

```
-gamma U_{-1}[f] U_1[] + U_{-1}[f] U_1[g]
```

```
swap[x_, y_] := USimp[(U_1[x] // qDelta_{1->1,2,3} // S_1) (U_4[y] // qDelta_{4->4,5,6}) // P_{1,4} // P_{3,6} // m_{5->1}
```

```
swap[g, h]
```

```
U_1[h] U_2[g]
```

```
swap[g, f]
```

```
-gamma U_1[f] U_2[] + U_1[f] U_2[g]
```

```
swap[G[1], f]
```

```
e^{-gamma} U_1[f] U_2[G[1]]
```

```
swap[e, h]
```

```
eta U_1[] U_2[e] + U_1[h] U_2[e]
```

```
swap[e, H[1]]
```

```
e^{gamma} U_1[H[1]] U_2[e]
```

```
swap[e, f]
```

```
U_1[H[1]] U_2[] + U_1[f] U_2[e] - U_1[] U_2[G[1]]
```