Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-04/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-05"];
```

## UEA` with provisional modification

This section is based on penseive://Projects/UEA/.

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≼ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. $PBWRule];
UUi_[1] := Ui[];
UUi_[x_^p_] := UUi @@ Table[x, {p}];
UUi_[ℰ_] := ℰ /. {
    U[xs__] :> Ui[xs],
    x_ /; MemberQ[$Basis, x] :> Ui[x]
   };
UUi_[x_, xs__] := UUt1[x] UUt2[xs] // Expand // mt1,t2→i;
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
ms__[0] = 0;
ms__[x_Plus] := ms /@ x;
ms__[sd_SeriesData] := MapAt[ms, sd, {3, All}];
mi_→j_[ℰ_] := ℰ /. Ui → Uj;
```

```
mi_,j_→k_[c_. Ui_[x___] Uj_[]] := c Uk[x];
mi_,j_→k_[c_. Ui_[] Uj_[y___]] := c Uk[y];
mi_,j_→k_[c_. Ui_[xx___, x_] Uj_[y_, yy___]] := If[x ≼ y,
   c Uk[xx, x, y, yy],
   ((Ui[xx] (Uj[y, x] + UUj[B[x, y]]) // Expand // mi,j→i) Uj[yy] // Expand // mi,j→k)
     c // USimp
  ];
```

```
Supp[ℰ_] := Union@Cases[{ℰ}, Ui_[___] :> i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = Supp[x] ⋂ Supp[y], σ, z},
   z = x; Do[z = m_{i→σ@i}[z], {i, is}];
   z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

```
𝔒[specs_, sd_SeriesData] := MapAt[𝔒[specs, #] &, sd, {3, All}];
𝔒[specs_, poly_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] :> Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[poly, vars] /. (ps_ → c_) :> c (
       specs /. MapThread[((#1 → _) :> #3^#2) &, {vars, ps, elems}] /. U_i_ :> UU_i
      )]
 ]
```

## The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\eta g}$ and to $H = e^{\gamma h}$, where $[g, e] = \gamma\, e$ and $[h, f] = -\eta\, f$.
Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2,\ G_1\, G_2,\ e_1 + G_1\, e_2,\ h_1 + h_2,\ H_1\, H_2,\ f_1\, H_2 + f_2)$.
Also, $(g, e)$ and $(h, f)$ are dual bases.

```
$Basis = {g, G, e, h, H, f}; $PBWRule = Thread[$Basis → Range@Length@$Basis];
```

```
𝔒[U₁[x → g], Normal@Series[e^(η x), {η, 0, 5}]]
```

$$U_1[\,] + \eta\, U_1[g] + \frac{1}{2}\eta^2\, U_1[g, g] + \frac{1}{6}\eta^3\, U_1[g, g, g] + \frac{1}{24}\eta^4\, U_1[g, g, g, g] + \frac{1}{120}\eta^5\, U_1[g, g, g, g, g]$$

```
B[g, e] = γ e; B[e, G] = (e^(-γ η) - 1) U[G, e]; B[g, G] = 0;
```

```
With[{G = 𝔒[U₁[x → g], Series[e^(η x), {η, 0, 5}]]}, UB[U₁[e], G] - (e^(-γ η) - 1) G ** U₁[e]]
```

$$O[\eta]^6$$

```
B[e, G]
```

$$\left(-1 + e^{-\gamma\,\eta}\right) U[G, e]$$

```
B[h, f] = -η f; B[f, H] = (e^(γ η) - 1) U[H, f]; B[f, H] = 0;
```

```
With[{H = 𝔒[U₁[x → h], Series[e^(γ x), {γ, 0, 5}]]}, UB[U₁[f], H] - (e^(γ η) - 1) H ** U₁[f]]
```

$$O[\gamma]^6$$

## The Co-Product and Co-Associativity

```
qΔ_→_,_[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
     U_i[] → U_j[] U_k[],
     U_i[g, xs___] :↦
      (USimp[(U_j[g] U_k[] + U_j[] U_k[g]) qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k),
     U_i[G, xs___] :↦ (USimp[U_j[G] U_k[G] qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k),
     U_i[e, xs___] :↦
      (USimp[(U_j[e] U_k[G] + U_j[] U_k[e]) qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k),
     U_i[h, xs___] :↦ (USimp[(U_j[h] U_k[] + U_j[] U_k[h]) qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j //
        m_k,tk→k),
     U_i[H, xs___] :↦ (USimp[U_j[H] U_k[H] qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k),
     U_i[f, xs___] :↦
      (USimp[(U_j[f] U_k[] + U_j[H] U_k[f]) qΔ_i→tj,tk[U_i[xs]]] // m_j,tj→j // m_k,tk→k)
    }]
```

```
U1[e] // qΔ1→1,2
```

$$U_1[\,]\,U_2[e] + U_1[e]\,U_2[G]$$

```
{lhs = U1[e] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[e] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
```

$$\{U_1[\,]\,U_2[\,]\,U_3[e] + U_1[\,]\,U_2[e]\,U_3[G] + U_1[e]\,U_2[G]\,U_3[G],\\ U_1[\,]\,U_2[\,]\,U_3[e] + U_1[\,]\,U_2[e]\,U_3[G] + U_1[e]\,U_2[G]\,U_3[G], \text{True}\}$$

```
U1[f] // qΔ1→1,2
```

$$U_1[f]\,U_2[\,] + U_1[H]\,U_2[f]$$

```
{lhs = U1[f] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[f] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
```

$$\{U_1[f]\,U_2[\,]\,U_3[\,] + U_1[H]\,U_2[f]\,U_3[\,] + U_1[H]\,U_2[H]\,U_3[f],\\ U_1[f]\,U_2[\,]\,U_3[\,] + U_1[H]\,U_2[f]\,U_3[\,] + U_1[H]\,U_2[H]\,U_3[f], \text{True}\}$$

## The Antipode

```
S[g] = -g; S[G] = Oops;
```

## The Pairing at Lie-Level and Compatibilities

```
P[U[], U[H...]] = P[U[G...], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
({P[U[g], U[h]] = 1   P[U[g], U[H]] = γ   P[U[g], U[f]] = 0
  P[U[G], U[h]] = η   P[U[G], U[H]] = ℯ^(ηγ)   P[U[G], U[f]] = 0
  P[U[e], U[h]] = 0   P[U[e], U[H]] = 0   P[U[e], U[f]] = 1});
```

```
P_i_,j_[ℰ_] := USimp[ℰ /. U_i[xs___] U_j[ys___] → P[U[xs], U[ys]]];
```

```
t = U_i[g] U_j[e] U_k[f];
{m_{i,j→i}[t] - m_{j,i→i}[t], qΔ_{k→k,l}[t] - qΔ_{k→l,k}[t]}
```

```
{γ U_i[e] U_k[f], U_i[g] U_j[e] U_k[f] U_l[] -
  U_i[g] U_j[e] U_k[] U_l[f] + U_i[g] U_j[e] U_k[H] U_l[f] - U_i[g] U_j[e] U_k[f] U_l[H]}
```

```
t = U_i[g] U_j[e] U_k[f];
{(m_{i,j→i}[t] - m_{j,i→i}[t]) // P_{i,k}, (qΔ_{k→k,l}[t] - qΔ_{k→l,k}[t]) // P_{i,k} // P_{j,l}}
```

```
{γ, γ}
```

```
Table[t = U_i[xi] U_j[xj] U_k[yk];
 {(m_{i,j→i}[t] - m_{j,i→i}[t]) // P_{i,k}, (qΔ_{k→k,l}[t] - qΔ_{k→l,k}[t]) // P_{i,k} // P_{j,l}},
 {xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {γ, γ}}}, {{{0, 0}, {-γ, -γ}}, {{0, 0}, {0, 0}}}}
```

```
Table[t = U_i[xi] U_k[yk] U_l[yl];
 {(qΔ_{i→i,j}[t] - qΔ_{i→j,i}[t]) // P_{i,k} // P_{j,l}, (m_{k,l→k}[t] - m_{l,k→k}[t]) // P_{i,k}},
 {xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}, {{{0, 0}, {-η, -η}}, {{η, η}, {0, 0}}}}
```

## General Pairings

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[x_, xs__], U[y_]] := P[U[x, xs], U[y]] =
   Module[{i, j, k, l}, USimp[U_i[x] U_j[xs] qΔ_{k→k,l}[U_k[y]]] // P_{i,k} // P_{j,l}];
P[U[xs__], U[y_, ys__]] := P[U[xs], U[y, ys]] =
   Module[{i, j, k, l}, USimp[qΔ_{i→i,j}[U_i[xs]] U_k[y] U_l[ys]] // P_{i,k} // P_{j,l}];
```

```
{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
```

```
{0, γ, 0}
```

```
P[U[e], U[f, f]]
```

```
0
```

```
Factor@{P[U[e, e], U[f, f]], P[U[e, e, e], U[f, f, f]],
  P[U[e, e, e, e], U[f, f, f, f]], P[U[e, e, e, e, e], U[f, f, f, f, f]]}
```

$$\left\{1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\right\}$$

```
Simplify@FunctionExpand@Table[QFactorial[n, e^{γ η}], {n, 5}]
```

$$\left\{1,\ 1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\right\}$$

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

```
120
```