Pensieve header: The double and meta-double of the 2D pencil; continues pensieve://2017-04/.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2017-05"];
```

UEA` and Provisional Extensions

```
Once[Get["UEA`"]]
```

```
UEA` does computations in general universal enveloping algebras and PBW algebras. It is in the
  public domain, available at http://drorbn.net/AcademicPensieve/Projects/UEA/. Dror Bar-Natan
  is committed to support it within reason until March 18, 2022. This is version 170503.
```

```
UEA` implements / extends {**, B, m, SetAlgebra, U, UB, UProducts, USimp, UU, $Basis, $PBWRule}.
```

```
UEA`SetAlgebra knows "sl2".
```

```
𝔒[specs_, sd_SeriesData] := MapAt[𝔒[specs, #] &, sd, {3, All}];
𝔒[specs_, poly_] := Module[{rules, vars, elems},
  rules = Union @@ Cases[{specs}, U_[u___] :> Cases[{u}, r_Rule], ∞];
  vars = First /@ rules; elems = Last /@ rules;
  USimp@Total[CoefficientRules[poly, vars] /. (ps_ → c_) :> c (
        specs /. MapThread[((#1 → _) :> #3^#2) &, {vars, ps, elems}] /. U_i_ :> UU_i
      )]
 ]
```

The 2D Lie BiAlgebra Pencil

I hope to stick to $G = e^{\eta g}$ and to $H = e^{\gamma h}$, where $[g, e] = \gamma\, e$ and $[h, f] = -\eta\, f$.
Also, $q\Delta_{12}(g, G, e, h, H, f) = (g_1 + g_2,\ G_1\, G_2,\ e_1 + G_1\, e_2,\ h_1 + h_2,\ H_1\, H_2,\ f_1\, H_2 + f_2)$.
Also, (g, e) and (h, f) are dual bases.

```
$Basis = {g, G, e, h, H, f}; $PBWRule = Thread[$Basis → Range@Length@$Basis];
```

```
𝔒[U1[x → g], Normal@Series[e^(η x), {η, 0, 5}]]
```

$$U_1[\,] + \eta\, U_1[g] + \frac{1}{2}\eta^2\, U_1[g, g] + \frac{1}{6}\eta^3\, U_1[g, g, g] + \frac{1}{24}\eta^4\, U_1[g, g, g, g] + \frac{1}{120}\eta^5\, U_1[g, g, g, g, g]$$

```
B[g, e] = γ e; B[e, G] = (e^(-γ η) - 1) U[G, e]; B[g, G] = 0;
```

```
With[{G = 𝔒[U1[x → g], Series[e^(η x), {η, 0, 5}]]}, UB[U1[e], G] - (e^(-γ η) - 1) G ** U1[e]]
```

$$O[\eta]^6$$

```
B[e, G]
```

$$\left(-1 + e^{-\gamma\,\eta}\right) U[G, e]$$

```
B[h, f] = -η f; B[f, H] = (e^(γ η) - 1) U[H, f]; B[f, H] = 0;
```

```
With[{H = 𝒪[U1[x → h], Series[e^(γ x), {γ, 0, 5}]]}, UB[U1[f], H] - (e^(γ η) - 1) H ** U1[f]]
```

```
O[γ]^6
```

The Co-Product and Co-Associativity

```
qΔi_→j_,k_[ℰ_] := USimp@Module[{tj, tk}, ℰ /. {
     Ui[] → Uj[] Uk[],
     Ui[g, xs___] :→
      (USimp[(Uj[g] Uk[] + Uj[] Uk[g]) qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k),
     Ui[G, xs___] :→ (USimp[Uj[G] Uk[G] qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k),
     Ui[e, xs___] :→
      (USimp[(Uj[e] Uk[G] + Uj[] Uk[e]) qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k),
     Ui[h, xs___] :→ (USimp[(Uj[h] Uk[] + Uj[] Uk[h]) qΔi→tj,tk[Ui[xs]]] // mj,tj→j //
        mk,tk→k),
     Ui[H, xs___] :→ (USimp[Uj[H] Uk[H] qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k),
     Ui[f, xs___] :→
      (USimp[(Uj[f] Uk[] + Uj[H] Uk[f]) qΔi→tj,tk[Ui[xs]]] // mj,tj→j // mk,tk→k)
    }]
```

```
U1[e] // qΔ1→1,2
```

```
U1[] U2[e] + U1[e] U2[G]
```

```
{lhs = U1[e] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[e] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
```

```
{U1[] U2[] U3[e] + U1[] U2[e] U3[G] + U1[e] U2[G] U3[G],
 U1[] U2[] U3[e] + U1[] U2[e] U3[G] + U1[e] U2[G] U3[G], True}
```

```
U1[f] // qΔ1→1,2
```

```
U1[f] U2[] + U1[H] U2[f]
```

```
{lhs = U1[f] // qΔ1→1,2 // qΔ2→2,3, rhs = U1[f] // qΔ1→1,3 // qΔ1→1,2, lhs == rhs}
```

```
{U1[f] U2[] U3[] + U1[H] U2[f] U3[] + U1[H] U2[H] U3[f],
 U1[f] U2[] U3[] + U1[H] U2[f] U3[] + U1[H] U2[H] U3[f], True}
```

The Antipode

```
S[g] = -g; S[G] = Oops;
```

The Pairing at Lie-Level and Compatibilities

```
P[U[], U[H...]] = P[U[G...], U[]] = 1;
P[U[], U[__]] = P[U[__], U[]] = 0;
({P[U[g], U[h]] = 1  P[U[g], U[H]] = γ  P[U[g], U[f]] = 0
  P[U[G], U[h]] = η  P[U[G], U[H]] = e^(η γ)  P[U[G], U[f]] = 0
  P[U[e], U[h]] = 0  P[U[e], U[H]] = 0  P[U[e], U[f]] = 1});
```

```
P_i_,j_[ℰ_] := USimp[ℰ /. U_i[xs___] U_j[ys___] → P[U[xs], U[ys]]];
```

```
t = U_i[g] U_j[e] U_k[f];
{m_i,j→i[t] - m_j,i→i[t], qΔ_k→k,l[t] - qΔ_k→l,k[t]}
```

```
{γ U_i[e] U_k[f], U_i[g] U_j[e] U_k[f] U_l[] -
  U_i[g] U_j[e] U_k[] U_l[f] + U_i[g] U_j[e] U_k[H] U_l[f] - U_i[g] U_j[e] U_k[f] U_l[H]}
```

```
t = U_i[g] U_j[e] U_k[f];
{(m_i,j→i[t] - m_j,i→i[t]) // P_i,k, (qΔ_k→k,l[t] - qΔ_k→l,k[t]) // P_i,k // P_j,l}
```

```
{γ, γ}
```

```
Table[t = U_i[xi] U_j[xj] U_k[yk];
 {(m_i,j→i[t] - m_j,i→i[t]) // P_i,k, (qΔ_k→k,l[t] - qΔ_k→l,k[t]) // P_i,k // P_j,l},
 {xi, {g, e}}, {xj, {g, e}}, {yk, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {γ, γ}}}, {{{0, 0}, {-γ, -γ}}, {{0, 0}, {0, 0}}}}
```

```
Table[t = U_i[xi] U_k[yk] U_l[yl];
 {(qΔ_i→i,j[t] - qΔ_i→j,i[t]) // P_i,k // P_j,l, (m_k,l→k[t] - m_l,k→k[t]) // P_i,k},
 {xi, {g, e}}, {yk, {h, f}}, {yl, {h, f}}]
```

```
{{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}, {{{0, 0}, {-η, -η}}, {{η, η}, {0, 0}}}}
```

General Pairings

The pairing sequence: ⟨one,one⟩ (above), ⟨many,one⟩, ⟨many,many⟩.

```
P[U[x_, xs__], U[y_]] := P[U[x, xs], U[y]] =
   Module[{i, j, k, l}, USimp[U_i[x] U_j[xs] qΔ_k→k,l[U_k[y]]] // P_i,k // P_j,l];
P[U[xs__], U[y_, ys__]] := P[U[xs], U[y, ys]] =
   Module[{i, j, k, l}, USimp[qΔ_i→i,j[U_i[xs]] U_k[y] U_l[ys]] // P_i,k // P_j,l];
```

```
{P[U[g, e], U[h]], P[U[g, e], U[f]], P[U[e, e], U[f]]}
```

```
{0, γ, 0}
```

```
P[U[e], U[f, f]]
```

```
0
```

```
Factor@{P[U[e, e], U[f, f]], P[U[e, e, e], U[f, f, f]],
  P[U[e, e, e, e], U[f, f, f, f]], P[U[e, e, e, e, e], U[f, f, f, f, f]]}
```

$$\left\{1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\right\}$$

```
Simplify@FunctionExpand@Table[QFactorial[n, e^γ η], {n, 5}]
```

$$\left\{1,\ 1+e^{\gamma\eta},\ \left(1+e^{\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right),\ \left(1+e^{\gamma\eta}\right)^2\left(1+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}\right)\left(1+e^{\gamma\eta}+e^{2\gamma\eta}+e^{3\gamma\eta}+e^{4\gamma\eta}\right)\right\}$$

```
P[U[g, g, g, g, g], U[h, h, h, h, h]]
```

```
120
```