Proving Quesne's formula

April 27, 2017 3:30 PM

(170427) Quesne@arXiv:math-ph/0305003: $\log \mathbb{e}_q^x = \sum_{k\geqslant 1} \frac{(1-q)^k x^k}{k(1-q^k)}$.
(170317) Wikipedia: q-derivative: $D_{q,x} f(x) = \frac{f(qx)-f(x)}{qx-x}$; has $D_{q,x}\mathbb{e}_q^x = \mathbb{e}_q^x$ (and $\mathbb{e}_q^0 = 1$); seek it and $\mathbb{e}_q^x$ and $xy = qyx$ in nature. Finds: $[l,e] = e \Rightarrow \mathbb{e}^{\alpha l} e = \mathbb{e}^{\alpha} e \mathbb{e}^{\alpha l}$.

Thm $\quad \mathbb{e}_q^x = \exp \sum_{k=1}^{\infty} \frac{(1-q^k)x^k}{k(1-q^k)} =: F_q(x)$ (is $F_q(x)$ the gen. function of anything useful?)

Pf NTS, $F_q(0)=1$ (trivial) & $D_{q,x}F_q(x) = F_q(x)$;
indeed,

$$D_{q,x}F_q(x) = \frac{\exp\sum_{k\geq 1}\frac{(1-q)^k q^k x^k}{k(1-q^k)} - \exp\sum_{k\geq 1}\frac{(1-q)^k x^k}{k(1-q^k)}}{qx-x}$$

$$= (qx-x)^{-1} \exp \frac{\sum \frac{(1-q)^k q^k x^k}{k(1-q^k)}}{\sum \frac{(1-q)^k x^k}{k(1-q^k)}} \quad ?$$

Proof in Quesne:arXiv:math-ph/0310038.

$$F_q(qx) - F_q(x) = x(q-1)F_q(x)$$