Basis and dual basis

Continues 2016-07

April 15, 2017 4:56 PM

$[l,f]=f \Rightarrow fl = lf - f = (l-1)f \Rightarrow f e^{\alpha l} = e^{\alpha(l-1)} f$

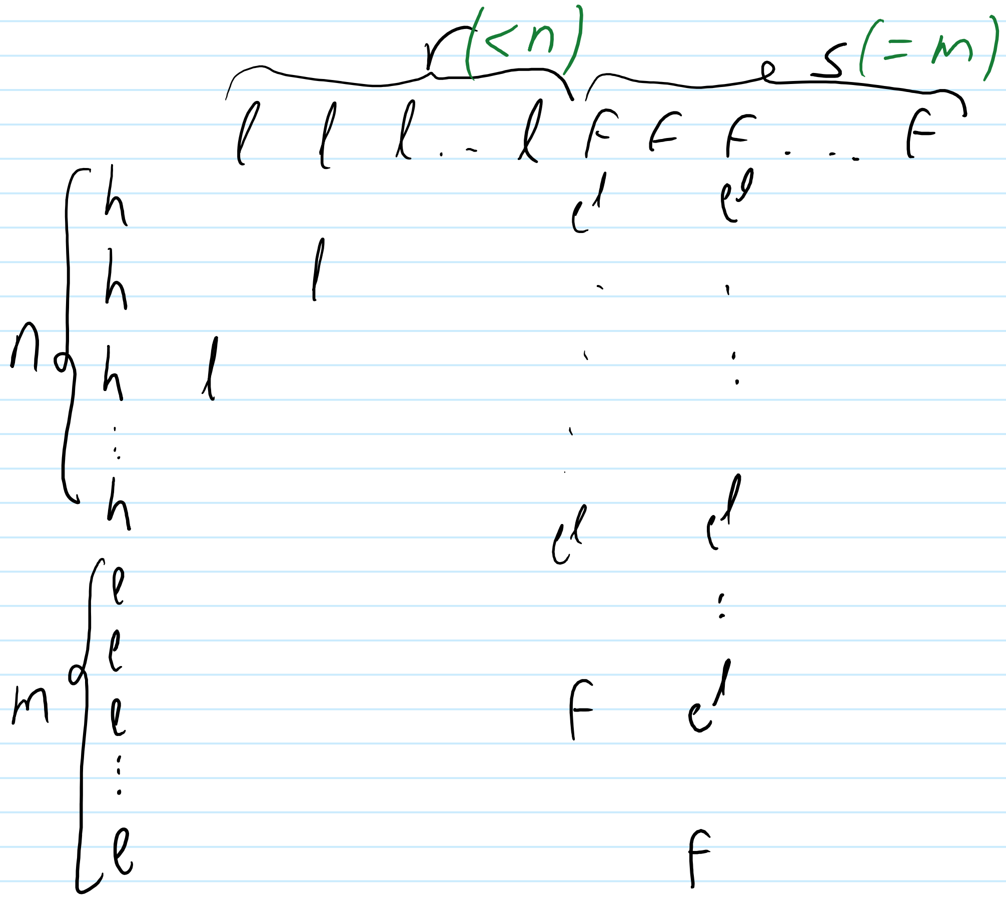
$\Delta(f) = f\otimes 1 + e^l \otimes f$ (h,e) & (l,f) are dual.

$\Delta^2(f) = \Delta f\otimes 1 + \Delta e^l \otimes f = f\otimes1\otimes1 + e^l\otimes f\otimes 1 + e^l\otimes e^l\otimes f$

$= f\otimes\Delta 1 + e^l\otimes \Delta f = f\otimes1\otimes1 + e^l\otimes f\otimes 1 + e^l\otimes e^l\otimes f$

What's the dual of $h^n e^m$?

$\langle h^n e^m, l^r f^s\rangle = \langle h^{\otimes n}\otimes e^{\otimes m}, \Delta^{n+m}(l^r f^s)\rangle$



$\neq \delta_{ms}\, m!\, \binom{n}{r}\cdot m^{n-r}$

From Majid::Primer:

Proposition 2.5 *$U_q(b_+)$ is dually paired with itself by*

$$\langle g,g\rangle = q,\quad \langle X,X\rangle = 1,\quad \langle X,g\rangle = \langle g,X\rangle = 0.$$

Proof We will see general methods for this kind of result later in the course. For the moment, it is a nice exercise directly from the definitions. Hint: first find that $f_{m,n}(g) \equiv \langle X^m g^n, g\rangle = \langle X^m, g\rangle\langle g^n, g\rangle = q^n\delta_{m,0}$ and $f_{m,n}(X) \equiv \langle X^m g^n, X\rangle = \langle X^m, X\rangle\langle g^n, 1\rangle = \delta_{m,1}$. Then the coproduct $\Delta(X^m g^n) = (\Delta X^m)(g^n \otimes g^n)$ given in the last lecture, and the axioms of a pairing, imply that

Page 6, see below.

$$f_{m,n}(hh') = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q f_{m-r,n+r}(h) f_{r,n}(h')$$

for all $h, h' \in U_q(b_+)$. This determines $f_{m,n}$ on products, which shows that $\langle\ ,\ \rangle$ is uniquely determined. We then define it on the basis $\{X^m g^n |\ n \in \mathbb{Z},\ m \in \mathbb{Z}_+\}$ of each copy of $U_q(b_+)$ (where $\mathbb{Z}_+$ includes 0), by the resulting formula for f_{mn}, and verify the duality pairing axioms on products and coproducts of basis elements. □

On page 6:

It is a nice exercise – we will prove it later in the course, but some readers may want to have fun doing it now – to show that

$$\Delta X^m = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q X^{m-r} g^r \otimes X^r$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!},\quad [r]_q! = [r]_q[r-1]_q \cdots [1]_q$$

are the q-binomial coefficients defined in terms of 'q-integers'

$$[r]_q = 1 + q + \cdots + q^{r-1} = \frac{1-q^r}{1-q}.$$