$$\text{Sum}\left[x^n \prod_{k=1}^{n} \frac{(q-1)}{q^k-1},\ \{n, 0, \infty\}\right]$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{-n} (-1+q)^n x^n}{\text{QPochhammer}[q, q, n]}$$

$$\text{t1} = \text{With}\left[\{n = 4\},\ x^n \prod_{k=1}^{n} \frac{(q-1)}{q^k-1}\ \text{/. } q \to e^{\epsilon}\right]$$

$$\frac{(-1+e^{\epsilon})^3 x^4}{(-1+e^{2\epsilon})(-1+e^{3\epsilon})(-1+e^{4\epsilon})}$$

$$\text{Series}[\text{t1}, \{\epsilon, 0, 3\}]$$

$$\frac{x^4}{24} - \frac{x^4 \epsilon}{8} + \frac{41 x^4 \epsilon^2}{288} - \frac{5 x^4 \epsilon^3}{96} + O[\epsilon]^4$$

$$\text{With}\left[\{n = 4\},\ \text{Normal@Series}\left[x^n \prod_{k=1}^{n} \frac{(q-1)}{q^k-1}\ \text{/. } q \to e^{\epsilon},\ \{\epsilon, 0, 3\}\right]\right]$$

$$\frac{x^4}{24} - \frac{x^4 \epsilon}{8} + \frac{41 x^4 \epsilon^2}{288} - \frac{5 x^4 \epsilon^3}{96}$$

$$\text{Sum}\left[\text{Normal@Series}\left[x^n \prod_{k=1}^{n} \frac{(q-1)}{q^k-1}\ \text{/. } q \to e^{\epsilon},\ \{\epsilon, 0, 0\}\right],\ \{n, 0, \infty\}\right]$$

Sum: Sum does not converge.

Sum: Sum does not converge.

$$\sum_{n=0}^{\infty} \frac{(-1)^{-n} x^n \epsilon^n}{\text{QPochhammer}[1, 1, n]}$$

$$\text{FullSimplify}\left[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1) x + 1) F[\epsilon, x]}{x}\ \text{/. } F[\epsilon\_, x\_] \mapsto \text{Sum}\left[x^n \prod_{k=1}^{n} \frac{(q-1)}{q^k-1}\ \text{/. } q \to e^{\epsilon},\ \{n, 0, \infty\}\right]\right]$$

$Aborted

$$\text{Simplify@Series}\left[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1) x + 1) F[\epsilon, x]}{x},\ \{\epsilon, 0, 3\}\right]$$

$$\left(-F[0, x] + F^{(0,1)}[0, x]\right) \epsilon + \frac{1}{2}\left(-F[0, x] + F^{(0,1)}[0, x] + x F^{(0,2)}[0, x] - 2 F^{(1,0)}[0, x] + 2 F^{(1,1)}[0, x]\right) \epsilon^2 + \frac{1}{6}\left(-F[0, x] + F^{(0,1)}[0, x] + 3 x F^{(0,2)}[0, x] + x^2 F^{(0,3)}[0, x] - 3 F^{(1,0)}[0, x] + 3 F^{(1,1)}[0, x] + 3 x F^{(1,2)}[0, x] - 3 F^{(2,0)}[0, x] + 3 F^{(2,1)}[0, x]\right) \epsilon^3 + O[\epsilon]^4$$

$$x^n \prod_{k=1}^{n} \text{Series}\left[\frac{(q-1)}{q^k-1} \text{ /. } q \to e^{\epsilon}, \{\epsilon, 0, 3\}\right]$$

$$x^n \prod_{k=1}^{n} \left(\frac{1}{k} + \left(-\frac{1}{2} + \frac{1}{2k}\right)\epsilon + \frac{(2-3k+k^2)\,\epsilon^2}{12k} + \frac{(1-2k+k^2)\,\epsilon^3}{24k} + O[\epsilon]^4\right)$$

$$\text{Simplify}\left[e^{-x-f[\epsilon,x]} \text{ Series}\left[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1)x+1)\,F[\epsilon, x]}{x} \text{ /. } F[\epsilon\_, x\_] :\to e^{x+f[\epsilon,x]}, \{\epsilon, 0, 3\}\right]\right]$$

$$f^{(0,1)}[0,x]\,\epsilon + \frac{1}{2}\left(x + (1+2x)\,f^{(0,1)}[0,x] + x\,f^{(0,1)}[0,x]^2 + x\,f^{(0,2)}[0,x] + 2\,f^{(1,1)}[0,x]\right)\epsilon^2 + \frac{1}{6}\left(3x + x^2 + 3x(1+x)\,f^{(0,1)}[0,x]^2 + x^2 f^{(0,1)}[0,x]^3 + 3x(1+x)\,f^{(0,2)}[0,x] + x^2 f^{(0,3)}[0,x] + 3\,f^{(1,1)}[0,x] + 6x\,f^{(1,1)}[0,x] + f^{(0,1)}[0,x]\left(1 + 6x + 3x^2 + 3x^2 f^{(0,2)}[0,x] + 6x\,f^{(1,1)}[0,x]\right) + 3x\,f^{(1,2)}[0,x] + 3\,f^{(2,1)}[0,x]\right)\epsilon^3 + O[\epsilon]^4$$

$$\text{Simplify}\left[\text{Series}\left[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1)x+1)\,F[\epsilon, x]}{x\,F[\epsilon, x]} \text{ /. } F[\epsilon\_, x\_] :\to e^{x+f[\epsilon,x]} \text{ /. } f[\epsilon\_, x\_] \to \epsilon\,\phi[\epsilon, x], \{\epsilon, 0, 3\}\right]\right]$$

$$\left(\frac{x}{2} + \phi^{(0,1)}[0,x]\right)\epsilon^2 + \frac{1}{6}\left(3x + x^2 + (3+6x)\,\phi^{(0,1)}[0,x] + 3x\,\phi^{(0,2)}[0,x] + 6\,\phi^{(1,1)}[0,x]\right)\epsilon^3 + O[\epsilon]^4$$

$$\text{Collect}\Big[\text{Normal@Series}\Big[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1)x+1)\,F[\epsilon, x]}{x\,F[\epsilon, x]} \text{ /. } F[\epsilon\_, x\_] :\to e^{x+f[\epsilon,x]} \text{ /. } f[\epsilon\_, x\_] \to \frac{-x^2}{4}\epsilon + \frac{x^3}{9}\epsilon^2 + \left(\frac{-x^4}{16} + \frac{x^2}{48}\right)\epsilon^3 + \left(\frac{x^5}{25} - \frac{x^3}{36}\right)\epsilon^4 + \left(\frac{-x^6}{36} + \frac{x^4}{32} - \frac{x^2}{480}\right)\epsilon^5 + \epsilon^6\,\phi[\epsilon, x], \{\epsilon, 0, 7\}\Big], \epsilon, \text{Expand}\Big]$$

$$\epsilon^7\left(-\frac{7x^2}{360} + \frac{x^4}{6} - \frac{x^6}{7} + \phi^{(0,1)}[0,x]\right)$$

$$\text{Collect}\Big[\text{Normal@Series}\Big[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1)x+1)\,F[\epsilon, x]}{x\,F[\epsilon, x]} \text{ /. } F[\epsilon\_, x\_] :\to e^{x+f[\epsilon,x]} \text{ /. } f[\epsilon\_, x\_] \to \frac{-x^2}{4}\epsilon + \frac{x^3}{9}\epsilon^2 + \left(\frac{-x^4}{16} + \frac{x^2}{48}\right)\epsilon^3 + \left(\frac{x^5}{25} - \frac{x^3}{36}\right)\epsilon^4 + \left(\frac{-x^6}{36} + \frac{x^4}{32} - \frac{x^2}{480}\right)\epsilon^5 + \epsilon^6\,\phi[\epsilon, x], \{\epsilon, 0, 7\}\Big], \epsilon, \text{Expand}\Big]$$

$$\epsilon^7\left(-\frac{7x^2}{360} + \frac{x^4}{6} - \frac{x^6}{7} + \phi^{(0,1)}[0,x]\right)$$

```
-Integrate[-7 x^2/360 + x^4/6 - x^6/7, x]
```

$$\frac{7x^3}{1080} - \frac{x^5}{30} + \frac{x^7}{49}$$

```
Collect[Normal@Series[(F[ϵ, ⅇ^ϵ x] - ((ⅇ^ϵ - 1) x + 1) F[ϵ, x])/(x F[ϵ, x]) /. F[ϵ_, x_] :> ⅇ^(x+f[ϵ,x]) /.
    f[ϵ_, x_] → -x^2/4 ϵ + x^3/9 ϵ^2 + (-x^4/16 + x^2/48) ϵ^3 + (x^5/25 - x^3/36) ϵ^4 +
      (-x^6/36 + x^4/32 - x^2/480) ϵ^5 + (7 x^3/1080 - x^5/30 + x^7/49) ϵ^6 + ϵ^7 ϕ[ϵ, x], {ϵ, 0, 8}],
  ϵ,
  Expand]
```

$$\epsilon^8\left(-\frac{17x}{40320} + \frac{17x^3}{320} - \frac{5x^5}{24} + \frac{x^7}{8} + \phi^{(0,1)}[0, x]\right)$$

```
-Integrate[-17 x/40320 + 17 x^3/320 - 5 x^5/24 + x^7/8, x]
```

$$\frac{17x^2}{80640} - \frac{17x^4}{1280} + \frac{5x^6}{144} - \frac{x^8}{64}$$

```
Collect[Normal@Series[(F[ϵ, ⅇ^ϵ x] - ((ⅇ^ϵ - 1) x + 1) F[ϵ, x])/(x F[ϵ, x]) /. F[ϵ_, x_] :> ⅇ^(x+f[ϵ,x]) /.
    f[ϵ_, x_] → -x^2/4 ϵ + x^3/9 ϵ^2 + (-x^4/16 + x^2/48) ϵ^3 + (x^5/25 - x^3/36) ϵ^4 + (-x^6/36 + x^4/32 - x^2/480) ϵ^5 +
      (7 x^3/1080 - x^5/30 + x^7/49) ϵ^6 + (17 x^2/80640 - 17 x^4/1280 + 5 x^6/144 - x^8/64) ϵ^7 + ϵ^8 ϕ[ϵ, x], {ϵ, 0, 9}],
  ϵ,
  Expand]
```

$$\epsilon^9\left(\frac{809x^2}{181440} - \frac{9x^4}{80} + \frac{x^6}{4} - \frac{x^8}{9} + \phi^{(0,1)}[0, x]\right)$$

```
-Integrate[809 x^2/181440 - 9 x^4/80 + x^6/4 - x^8/9, x]
```

$$-\frac{809x^3}{544320} + \frac{9x^5}{400} - \frac{x^7}{28} + \frac{x^9}{81}$$

$$\text{Collect}\Big[\text{Normal@Series}\Big[\frac{F[\epsilon, e^{\epsilon} x] - ((e^{\epsilon}-1)x+1)F[\epsilon,x]}{x F[\epsilon,x]} \text{ /. } F[\epsilon\_, x\_] :\to e^{x+f[\epsilon,x]} \text{ /. } f[\epsilon\_, x\_] \to \frac{-x^2}{4}\epsilon + \frac{x^3}{9}\epsilon^2 + \left(\frac{-x^4}{16}+\frac{x^2}{48}\right)\epsilon^3 + \left(\frac{x^5}{25}-\frac{x^3}{36}\right)\epsilon^4 + \left(\frac{-x^6}{36}+\frac{x^4}{32}-\frac{x^2}{480}\right)\epsilon^5 + \left(\frac{7x^3}{1080}-\frac{x^5}{30}+\frac{x^7}{49}\right)\epsilon^6 + \left(\frac{17x^2}{80640}-\frac{17x^4}{1280}+\frac{5x^6}{144}-\frac{x^8}{64}\right)\epsilon^7 + \left(-\frac{809x^3}{544320}+\frac{9x^5}{400}-\frac{x^7}{28}+\frac{x^9}{81}\right)\epsilon^8 + \epsilon^9\phi[\epsilon,x], \{\epsilon, 0, 10\}\Big], \epsilon, \text{Expand}\Big]$$

$$\epsilon^{10}\left(\frac{31x}{725760}-\frac{527x^3}{24192}+\frac{59x^5}{288}-\frac{7x^7}{24}+\frac{x^9}{10}+\phi^{(0,1)}[0,x]\right)$$

$$-\text{Integrate}\left[\frac{31x}{725760}-\frac{527x^3}{24192}+\frac{59x^5}{288}-\frac{7x^7}{24}+\frac{x^9}{10}, x\right]$$

$$-\frac{31x^2}{1451520}+\frac{527x^4}{96768}-\frac{59x^6}{1728}+\frac{7x^8}{192}-\frac{x^{10}}{100}$$

$$\text{Factor}\left[-\frac{31x^2}{1451520}+\frac{527x^4}{96768}-\frac{59x^6}{1728}+\frac{7x^8}{192}-\frac{x^{10}}{100}\right]$$

$$-\frac{x^2\left(155-39525x^2+247800x^4-264600x^6+72576x^8\right)}{7257600}$$

Comparing with Quesne’s arXiv:math-ph/0305003:

$$c_{k\_}[q\_] := \frac{(1-q)^k}{k(1-q^k)};$$

$$\text{With}\Big[\{n=5\}, \text{Collect}\Big[e^{-x}\,\text{Normal@Series}\big[\text{Exp}\big[\text{Sum}\big[c_k[e^{\epsilon}]\,x^k, \{k, 1, n+1\}\big]\big], \{\epsilon, 0, n\}\big], \epsilon, \text{Expand}\Big]\Big]$$

$$1-\frac{x^2\epsilon}{4}+\left(\frac{x^3}{9}+\frac{x^4}{32}\right)\epsilon^2+\left(\frac{x^2}{48}-\frac{x^4}{16}-\frac{x^5}{36}-\frac{x^6}{384}\right)\epsilon^3+\left(-\frac{x^3}{36}-\frac{x^4}{192}+\frac{x^5}{25}+\frac{113x^6}{5184}+\frac{x^7}{288}+\frac{x^8}{6144}\right)\epsilon^4+\left(-\frac{x^2}{480}+\frac{x^4}{32}+\frac{x^5}{108}-\frac{125x^6}{4608}-\frac{61x^7}{3600}-\frac{145x^8}{41472}-\frac{x^9}{3456}-\frac{x^{10}}{122880}\right)\epsilon^5$$

$$c_{k\_}[q\_] := \frac{(1-q)^k}{k\,(1-q^k)};$$

```
With[{n = 9}, Collect[
  Normal@Series[Sum[c_k[ⅇ^ϵ] x^k, {k, 1, n + 1}], {ϵ, 0, n}],
  ϵ, Expand
 ]]
```

$$x - \frac{x^2 \epsilon}{4} + \frac{x^3 \epsilon^2}{9} + \left(\frac{x^2}{48} - \frac{x^4}{16}\right)\epsilon^3 + \left(-\frac{x^3}{36} + \frac{x^5}{25}\right)\epsilon^4 + \left(-\frac{x^2}{480} + \frac{x^4}{32} - \frac{x^6}{36}\right)\epsilon^5 + \left(\frac{7x^3}{1080} - \frac{x^5}{30} + \frac{x^7}{49}\right)\epsilon^6 + \left(\frac{17x^2}{80640} - \frac{17x^4}{1280} + \frac{5x^6}{144} - \frac{x^8}{64}\right)\epsilon^7 + \left(-\frac{809x^3}{544320} + \frac{9x^5}{400} - \frac{x^7}{28} + \frac{x^9}{81}\right)\epsilon^8 + \left(-\frac{31x^2}{1451520} + \frac{527x^4}{96768} - \frac{59x^6}{1728} + \frac{7x^8}{192} - \frac{x^{10}}{100}\right)\epsilon^9$$