```
MatrixExp[a {{0, 0}, {1, 0}}].MatrixExp[b {{0, 1}, {0, 0}}] // MatrixForm
```

$$\begin{pmatrix} 1 & b \\ a & 1+a\,b \end{pmatrix}$$

```
MatrixExp[a {{0, 0}, {1, 0}}].MatrixExp[b {{0, 1}, {0, 0}}] // MatrixLog // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{-\sqrt{4+a\,b}\,\text{Log}[4]+\left(\sqrt{a}\,\sqrt{b}+\sqrt{4+a\,b}\right)\text{Log}\left[2+a\,b-\sqrt{a}\,\sqrt{b}\,\sqrt{4+a\,b}\right]+\left(-\sqrt{a}\,\sqrt{b}+\sqrt{4+a\,b}\right)\text{Log}\left[2+a\,b+\sqrt{a}\,\sqrt{b}\,\sqrt{4+a\,b}\right]}{2\sqrt{4+a\,b}} & \\ \frac{\sqrt{a}\left(-\text{Log}\left[2+a\,b-\sqrt{a}\,\sqrt{b}\,\sqrt{4+a\,b}\right]+\text{Log}\left[2+a\,b+\sqrt{a}\,\sqrt{b}\,\sqrt{4+a\,b}\right]\right)}{\sqrt{b}\,\sqrt{4+a\,b}} & \frac{-\sqrt{4+a\,b}\,\text{Log}[4]+\left(-\right.}{} \end{pmatrix}$$

```
eqn = MatrixExp[a {{0, 0}, {1, 0}}].MatrixExp[b {{0, 1}, {0, 0}}] ==
  MatrixExp[β {{0, 1}, {0, 0}}].MatrixExp[α {{0, 0}, {1, 0}}].MatrixExp[γ {{1, 0}, {0, -1}}]
```

$$\{\{1, b\}, \{a, 1+a\,b\}\} == \left\{\left\{e^{\gamma}\left(1+\alpha\,\beta\right), e^{-\gamma}\,\beta\right\}, \left\{e^{\gamma}\,\alpha, e^{-\gamma}\right\}\right\}$$

```
Solve[Thread[Flatten /@ eqn], {α, β, γ}]
```

$$\left\{\left\{\alpha \to a\left(1+a\,b\right), \beta \to \frac{b}{1+a\,b}, \right.\right.$$
$$\left.\left.\gamma \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1]+\text{Log}\left[\frac{1}{1+a\,b}\right], C[1] \in \text{Integers}\right]\right\}\right\}$$

```
Thread[Flatten /@ eqn] /. {α → a (1 + a b), β → b/(1 + a b), γ → Log[1/(1 + a b)]}
```

$$\{\text{True}, \text{True}, \text{True}, \text{True}\}$$

```
ρh = {{0, 0, -1}, {0, 0, 0}, {0, 0, 0}};
ρe = {{0, 0, 0}, {0, 0, 1}, {0, 0, 0}}; ρl = {{0, 0, 0}, {0, 1, 0}, {0, 0, 0}}; ρf = {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}}; ρ0 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}};
```

```
eqn2 = MatrixExp[a ρf].MatrixExp[b ρe] == MatrixExp[β ρe].MatrixExp[α ρf].MatrixExp[γ ρh]
```

$$\{\{1, a, a\,b\}, \{0, 1, b\}, \{0, 0, 1\}\} == \{\{1, \alpha, -\gamma\}, \{0, 1, \beta\}, \{0, 0, 1\}\}$$

```
Solve[Thread[Flatten /@ eqn2], {α, β, γ}]
```

$$\{\{\alpha \to a, \beta \to b, \gamma \to -a\,b\}\}$$