

Pensieve header: Implementing $\mathcal{U}(sl_2)$. Borrows from pensieve://2017-02/nb/gline.pdf.

```

B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : (c | e | f | h), y_] := Expand[c B[x, y]];
B[y_, c_ * x : (c | e | f | h)] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];

```

```

B[h, e] = 2 e; B[h, f] = -2 f; B[e, f] = h; B[c, _] = 0;
Basis[] := {c, e, f, h};

```

Basis[]

{c, e, f, h}

Table[{x, y} → B[x, y], {x, Basis[]}, {y, Basis[]} // MatrixForm

$$\begin{pmatrix} \{c, c\} \rightarrow 0 & \{c, e\} \rightarrow 0 & \{c, f\} \rightarrow 0 & \{c, h\} \rightarrow 0 \\ \{e, c\} \rightarrow 0 & \{e, e\} \rightarrow 0 & \{e, f\} \rightarrow h & \{e, h\} \rightarrow -2 e \\ \{f, c\} \rightarrow 0 & \{f, e\} \rightarrow -h & \{f, f\} \rightarrow 0 & \{f, h\} \rightarrow 2 f \\ \{h, c\} \rightarrow 0 & \{h, e\} \rightarrow 2 e & \{h, f\} \rightarrow -2 f & \{h, h\} \rightarrow 0 \end{pmatrix}$$

Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[Basis[], 2]}

{0}

DeleteCases[Table[

```

  ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
  {t, Tuples[Basis[], 3]}
], _ → 0]

```

{}

```

PBWRule = {};

```

```

x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. PBWRule];

```

```

UUi[ε_] := ε /. (x : c | e | f | h) ⇒ Ui[x];

```

```

Simp[ε_] := Collect[ε, Times[U[___] ..], Expand];

```

```

Simp[ε_] := Expand[ε];

```

a U₁[a, b] U₂[c, d] + b U₁[a, b] U₂[c, d] + c U₁[a, a] U₂[c, d] + a U₁[a, c] + b U₁[a, c] // Simp

a U₁[a, c] + b U₁[a, c] + c U₁[a, a] U₂[c, d] + a U₁[a, b] U₂[c, d] + b U₁[a, b] U₂[c, d]

e + f // UU₃

U₃[e] + U₃[f]

```

ms[0] = 0; ms[x_Plus] := ms /@ x;

```

```

mi→j[ε_] := ε /. Ui → Uj;

```

```
m1→4[U1[e, f] U2[e, f] U3[f]]
```

```
U2[e, f] U3[f] U4[e, f]
```

```
mi,j→k[c_. Ui[x___] Uj[]] := c Uk[x];
mi,j→k[c_. Ui[] Uj[y___]] := c Uk[y];
mi,j→k[c_. Ui[xx___, x_] Uj[y_, yy___]] := If[x ≤ y,
  c Uk[xx, x, y, yy],
  ((Ui[xx] (Uj[y, x] + UUj[B[x, y]])) // Expand // mi,j→i) Uj[yy] // Expand // mi,j→k)
  c // Simp
];
```

```
c U1[e, f] U2[e, e, f] + (a + b) U1[h] U2[f, f] // m1,2→3
```

```
-4 a U3[f, f] - 4 b U3[f, f] + 2 c U3[e, e, f] +
```

```
a U3[f, f, h] + b U3[f, f, h] - 2 c U3[e, e, f, h] + c U3[e, e, e, f, f]
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] := Sort@Flatten@
  Table[(Ui@@@Subsets[Basis[], {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
UProducts[{1}, 1]
```

```
{U1[c], U1[e], U1[f], U1[h]}
```

```
Union[(u ↦ m1,3→1[m1,2→1[u]] - m1,2→1[m2,3→2[u]]) /@ UProducts[{1, 2, 3}, 3]]
```

```
{0}
```

```
S[ $\mathcal{E}$ ] := Union@Cases[{\mathcal{E}}, Ui[___] ↦ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = S[x] ∩ S[y], σ, z},
  z = x; Do[z = mi→σ@i[z], {i, is}];
  z = Expand[y z]; Do[z = mσ@i, i→i[z], {i, is}]; z];
UB[x_, y_] := Simp[x ** y - y ** x];
```

```
{x = RandomChoice[UProducts[{1, 2}, 3]], y = RandomChoice[UProducts[{1, 2, 3}, 3]], x ** y}
```

```
{U1[h] U2[c, f], U1[e] U2[] U3[c, f], 2 U1[e] U2[c, f] U3[c, f] + U1[e, h] U2[c, f] U3[c, f]}
```

$$r_{i,j} := U_i[e] U_j[f] + \frac{1}{4} U_i[h] U_j[h] + \alpha U_i[h] U_j[c] - \alpha U_i[c] U_j[h];$$

```
UB[r1,2, r1,3] + UB[r1,2, r2,3] + UB[r1,3, r2,3]
```

```
0
```