Pensieve header: The true ${\mathfrak g}^\epsilon$ within sl_2 and without; old technology UEA.

$\mathfrak{tg}^\epsilon$ within sl_2

```
ρe = (0 1; 0 0); ρf = (0 0; ϵ 0); ρh = (ϵ 0; 0 -ϵ); ρg = (1 0; 0 -1); ρ0 = (0 0; 0 0);
MB[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
Simplify@{MB[ρg, ρe] == 2 ρe, MB[ρg, ρf] == -2 ρf, MB[ρe, ρf] == ρh, ρh - ϵ ρg == ρ0}
```

{True, True, True, True}

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

```
ρb = (-1 0; 0 -1); gl2rule = {b → ρb, c → (ϵ^-1 ρb - ρg) / 2, u → -ρf, w → ρe};
Simplify[{MB[w, c] == w, MB[c, u] == u, MB[u, w] == b - 2 ϵ c} /. gl2rule]
```

{True, True, True}

```
MatrixForm /@ Simplify /@ ({w, u, b, c} /. gl2rule)
```

$$\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\-\epsilon&0\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}-\frac{1+\epsilon}{2\epsilon}&0\\0&\frac{-1+\epsilon}{2\epsilon}\end{pmatrix}\right\}$$

```
(b - ϵ c) ⊗ c + u ⊗ w /. gl2rule /. a_ ⊗ b_ :→ MatrixForm /@ Simplify /@ (a ⊗ b)
```

$$\begin{pmatrix}0&0\\-\epsilon&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}\frac12(-1+\epsilon)&0\\0&\frac12(-1-\epsilon)\end{pmatrix}\otimes\begin{pmatrix}-\frac{1+\epsilon}{2\epsilon}&0\\0&\frac{-1+\epsilon}{2\epsilon}\end{pmatrix}$$

```
(b - ϵ c) ⊗ (-c) + (-u) ⊗ w /. {c → (ϵ^-1 b - g) / 2, u → -f, w → e} /. a_ ⊗ b_ :→ Simplify /@ (a ⊗ b)
```

$$f\otimes e+\left(\frac12(b+g\epsilon)\right)\otimes\left(\frac12\left(g-\frac{b}{\epsilon}\right)\right)$$

Implementing $\mathfrak{tg}^\epsilon=\langle b,e,g,f\rangle/[g,e]=2e,\ [g,f]=-2f,\ [e,f]=b+\epsilon g,\ [b,*]=0)$

```
PBWRule = {e → 1, g → 2, f → 3};
B[U@g, U@e] = 2 U@e; B[U@g, U@f] = -2 U@f; B[U@e, U@f] = b U[] + ϵ U[g];
```

```
$TD = 3; ℏ /: ℏ^d_· /; d > $TD := 0;
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. PBWRule];
Simp[ℰ_] := Collect[ℰ, _U, Expand];
```

```
U_i_[ℰ_] := ℰ /. {b → b_i, t → t_i, u_U :→ Replace[u, x_ :→ x_i, 1]};
B[U[(x_)_i_], U[(y_)_i_]] := B[U[x_i], U[y_i]] = U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, x_] = 0;
B[U[y_], U[x_]] := B[U[y], U[x]] = Simp[-B[U[x], U[y]]];
B[x_, y_] := x ** y - y ** x;
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
```

```
U[xx___, x_] ** U[y_, yy___] := If[x ≤ y, U[xx, x, y, yy], U@xx ** (U@y ** U@x + B[U@x, U@y]) ** U@yy];
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[];
UU[l_, r___] := U[l] ** UU[r];
```

```
UProducts[{}, 0] = {UU[]};
UProducts[{}, n_Integer] /; n > 0 = {};
UProducts[{x_, xs___}, n_Integer] :=
  Sort@Flatten@Table[UU[x^k] ** u, {k, 0, n}, {u, UProducts[{xs}, n - k]}];
UProducts[xs_List, k_Integer, n_Integer] := UProducts[Flatten@Table[x_j, {x, xs}, {j, k}], n];
UProducts[any__, {n_}] := Flatten@Table[UProducts[any, k], {k, 0, n}];
```

```
m[i_, j_, k_][ℰ_] := Simp[ℰ /. {
    u_U :> UU @@ Join[DeleteCases[u, x_i|j], U @@ Cases[u, x_i :> x_k], U @@ Cases[u, x_j :> x_k]],
    b_i|j → b_k}]
```

```
$Basis = {U@e, U@g, U@f};
```

```
Table[{x, y} → B[x, y], {x, $Basis}, {y, $Basis}] // MatrixForm
```

{U[e], U[e]} → 0	{U[e], U[g]} → −2 U[e]	{U[e], U[f]} → b U[] + ∈ U[g]
{U[g], U[e]} → 2 U[e]	{U[g], U[g]} → 0	{U[g], U[f]} → −2 U[f]
{U[f], U[e]} → −b U[] − ∈ U[g]	{U[f], U[g]} → 2 U[f]	{U[f], U[f]} → 0

```
Module[{x, y}, Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[$Basis, 2]}]]
```

{0}

```
Module[{x, y, z}, DeleteCases[Table[
    ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
    {t, Tuples[$Basis, 3]}
  ], _ → 0]]
```

{}

$$f\otimes e + \left(\frac{1}{2}\left(b + g\,\epsilon\right)\right)\otimes\left(\frac{1}{2}\left(g - \frac{b}{\epsilon}\right)\right)$$

$$f\otimes e + \left(\frac{1}{2}\left(b + g\,\epsilon\right)\right)\otimes\left(\frac{1}{2}\left(g - \frac{b}{\epsilon}\right)\right)$$

$$r_{i\_,j\_} := \text{USimp}\left[U_i[f]\,U_j[e] + \frac{1}{4}\left(-\epsilon^{-1}\,\delta\,U_i[b]\,U_j[b] + 2\,U_i[b]\,U_j[g] + \epsilon\,U_i[g]\,U_j[g]\right) + \alpha\left(U_i[b]\,U_j[g] - U_i[g]\,U_j[b]\right)\right];$$

$r_{1,2}$

$$-\frac{\delta\,U_1[b]\,U_2[b]}{4\,\epsilon} - \alpha\,U_1[g]\,U_2[b] + U_1[f]\,U_2[e] + \frac{1}{2}U_1[b]\,U_2[g] + \alpha\,U_1[b]\,U_2[g] + \frac{1}{4}\epsilon\,U_1[g]\,U_2[g]$$

$r_{1,2}$ /. $\{\delta \to 0, \alpha \to \frac{-1}{4}\}$

$\frac{1}{4} U_1[g] U_2[b] + U_1[f] U_2[e] + \frac{1}{4} U_1[b] U_2[g] + \frac{1}{4} \in U_1[g] U_2[g]$

UB[$r_{1,2}$, $r_{1,3}$] + UB[$r_{1,2}$, $r_{2,3}$] + UB[$r_{1,3}$, $r_{2,3}$] // USimp

0

UB[$r_{1,2}$, $r_{1,3}$]

$-2\alpha U_1[f] U_2[e] U_3[b] + 2\alpha U_1[f] U_2[b] U_3[e] - \frac{1}{2} \in U_1[f] U_2[g] U_3[e] + \frac{1}{2} \in U_1[f] U_2[e] U_3[g]$

UB[$r_{1,2}$, $r_{2,3}$]

$2\alpha U_1[f] U_2[e] U_3[b] + U_1[f] U_2[b] U_3[e] - U_1[b] U_2[f] U_3[e] - 2\alpha U_1[b] U_2[f] U_3[e] - \frac{1}{2} \in U_1[g] U_2[f] U_3[e] + \in U_1[f] U_2[g] U_3[e] - \frac{1}{2} \in U_1[f] U_2[e] U_3[g]$

UB[$r_{1,3}$, $r_{2,3}$]

$-U_1[f] U_2[b] U_3[e] - 2\alpha U_1[f] U_2[b] U_3[e] + U_1[b] U_2[f] U_3[e] + 2\alpha U_1[b] U_2[f] U_3[e] + \frac{1}{2} \in U_1[g] U_2[f] U_3[e] - \frac{1}{2} \in U_1[f] U_2[g] U_3[e]$