Pensieve header: The true ${\mathfrak g}^\epsilon$ within sl_2 and without.

Implementing general universal enveloping algebras

```
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x_, y_] /; MemberQ[$Basis, x] := Expand[c B[x, y]];
B[y_, c_ * x_] /; MemberQ[$Basis, x] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
B[x_, x_] = 0;
B[y_, x_] := Expand[-B[x, y]];
```

```
x_ ≤ y_ := OrderedQ[{x, y} /. $PBWRule]; x_ < y_ := ! OrderedQ[{y, x} /. $PBWRule];
UU_i_[ℰ_] := ℰ /. x_ /; MemberQ[$Basis, x] ⧴ U_i[x];
USimp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
USimp[ℰ_] := Expand[ℰ];
```

```
m_s__[0] = 0; m_s__[x_Plus] := m_s /@ x;
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≤ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j→i) U_j[yy] // Expand // m_i,j→k) c // USimp
  ];
```

```
UProducts[{}, 0] = {1}; UProducts[{}, d_Integer] /; d > 0 = {};
UProducts[{i_, is___}, d_Integer] :=
  Sort@Flatten@Table[(U_i @@@ Subsets[$Basis, {j}]) u, {j, 0, d}, {u, UProducts[{is}, d - j]}];
```

```
S[ℰ_] := Union@Cases[{ℰ}, U_i_[___] ⧴ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = S[x] ⋂ S[y], σ, z},
   z = x; Do[z = m_i→σ@i[z], {i, is}];
   z = Expand[y z]; Do[z = m_σ@i,i→i[z], {i, is}]; z];
UB[x_, y_] := USimp[x ** y - y ** x];
```

tg^ϵ within sl_2

```
ρe = (0 1; 0 0); ρf = (0 0; ϵ 0); ρh = (ϵ 0; 0 -ϵ); ρg = (1 0; 0 -1); ρ0 = (0 0; 0 0);
MB[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
Simplify@{MB[ρg, ρe] == 2 ρe, MB[ρg, ρf] == -2 ρf, MB[ρe, ρf] == ρh, ρh - ϵ ρg == ρ0}
```

```
{True, True, True, True}
```

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

```
ρb = ( -1  0 ; 0  -1 ); gl2rule = {b → ρb, c → (ϵ^-1 ρb - ρg) / 2, u → -ρf, w → ρe};
Simplify[{MB[w, c] == w, MB[c, u] == u, MB[u, w] == b - 2 ϵ c} /. gl2rule]
```

{True, True, True}

```
MatrixForm /@ Simplify /@ ({w, u, b, c} /. gl2rule)
```

$\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\-\epsilon&0\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}, \begin{pmatrix}-\frac{1+\epsilon}{2\epsilon}&0\\0&\frac{-1+\epsilon}{2\epsilon}\end{pmatrix}\right\}$

```
(b - ϵ c) ⊗ c + u ⊗ w /. gl2rule /. a_ ⊗ b_ :→ MatrixForm /@ Simplify /@ (a ⊗ b)
```

$\begin{pmatrix}0&0\\-\epsilon&0\end{pmatrix} \otimes \begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}(-1+\epsilon)&0\\0&\frac{1}{2}(-1-\epsilon)\end{pmatrix} \otimes \begin{pmatrix}-\frac{1+\epsilon}{2\epsilon}&0\\0&\frac{-1+\epsilon}{2\epsilon}\end{pmatrix}$

```
(b - ϵ c) ⊗ (-c) + (-u) ⊗ w /. {c → (ϵ^-1 b - g) / 2, u → -f, w → e} /. a_ ⊗ b_ :→ Simplify /@ (a ⊗ b)
```

$f \otimes e + \left(\frac{1}{2}(b + g\epsilon)\right) \otimes \left(\frac{1}{2}\left(g - \frac{b}{\epsilon}\right)\right)$

Implementing $t\mathfrak{g}^\epsilon = \langle b, e, g, f\rangle / [g, e] = 2e,\ [g, f] = -2f,\ [e, f] = b + \epsilon g,\ [b, *] = 0)$

```
B[g, e] = 2 e; B[g, f] = -2 f; B[e, f] = b + ϵ g;  B[b, _] = 0;
$Basis = {b, e, g, f};
$PBWRule = {b → 1, e → 2, g → 3, f → 4};
```

```
Table[{x, y} → B[x, y], {x, $Basis}, {y, $Basis}] // MatrixForm
```

{b, b} → 0	{b, e} → 0	{b, g} → 0	{b, f} → 0
{e, b} → 0	{e, e} → 0	{e, g} → -2 e	{e, f} → b + g ϵ
{g, b} → 0	{g, e} → 2 e	{g, g} → 0	{g, f} → -2 f
{f, b} → 0	{f, e} → -b - g ϵ	{f, g} → 2 f	{f, f} → 0

```
Module[{x, y}, Union@Table[{x, y} = t; B[x, y] + B[y, x], {t, Tuples[$Basis, 2]}]]
```

{0}

```
Module[{x, y, z}, DeleteCases[Table[
    ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
    {t, Tuples[$Basis, 3]}
  ], _ → 0]]
```

{}

```
Union[(u ↦ m_{1,3→1}[m_{1,2→1}[u]] - m_{1,2→1}[m_{2,3→2}[u]]) /@ UProducts[{1, 2, 3, 4}, 4]]
```

{0}

$f \otimes e + \left(\frac{1}{2}(b + g\epsilon)\right) \otimes \left(\frac{1}{2}\left(g - \frac{b}{\epsilon}\right)\right)$

$f \otimes e + \left(\frac{1}{2}(b + g\epsilon)\right) \otimes \left(\frac{1}{2}\left(g - \frac{b}{\epsilon}\right)\right)$

$r_{i\_,j\_}$:= USimp$\left[U_i[f]\,U_j[e] + \frac{1}{4}\left(-\epsilon^{-1}\,\delta\,U_i[b]\,U_j[b] + 2\,U_i[b]\,U_j[g] + \epsilon\,U_i[g]\,U_j[g]\right) + \alpha\left(U_i[b]\,U_j[g] - U_i[g]\,U_j[b]\right)\right]$;

$r_{1,2}$

$$-\frac{\delta\,U_1[b]\,U_2[b]}{4\,\epsilon} - \alpha\,U_1[g]\,U_2[b] + U_1[f]\,U_2[e] + \frac{1}{2}\,U_1[b]\,U_2[g] + \alpha\,U_1[b]\,U_2[g] + \frac{1}{4}\,\epsilon\,U_1[g]\,U_2[g]$$

$r_{1,2}$ /. $\left\{\delta \to 0,\ \alpha \to \frac{-1}{4}\right\}$

$$\frac{1}{4}\,U_1[g]\,U_2[b] + U_1[f]\,U_2[e] + \frac{1}{4}\,U_1[b]\,U_2[g] + \frac{1}{4}\,\epsilon\,U_1[g]\,U_2[g]$$

$r_{1,2}$ /. $\left\{\delta \to 0,\ \alpha \to \frac{-1}{2}\right\}$

$$\frac{1}{2}\,U_1[g]\,U_2[b] + U_1[f]\,U_2[e] + \frac{1}{4}\,\epsilon\,U_1[g]\,U_2[g]$$

UB$[r_{1,2}, r_{1,3}]$ + UB$[r_{1,2}, r_{2,3}]$ + UB$[r_{1,3}, r_{2,3}]$ // USimp

0

UB$[r_{1,2}, r_{1,3}]$

$$-2\,\alpha\,U_1[f]\,U_2[e]\,U_3[b] + 2\,\alpha\,U_1[f]\,U_2[b]\,U_3[e] - \frac{1}{2}\,\epsilon\,U_1[f]\,U_2[g]\,U_3[e] + \frac{1}{2}\,\epsilon\,U_1[f]\,U_2[e]\,U_3[g]$$

UB$[r_{1,2}, r_{2,3}]$

$$2\,\alpha\,U_1[f]\,U_2[e]\,U_3[b] + U_1[f]\,U_2[b]\,U_3[e] - U_1[b]\,U_2[f]\,U_3[e] - 2\,\alpha\,U_1[b]\,U_2[f]\,U_3[e] - \frac{1}{2}\,\epsilon\,U_1[g]\,U_2[f]\,U_3[e] + \epsilon\,U_1[f]\,U_2[g]\,U_3[e] - \frac{1}{2}\,\epsilon\,U_1[f]\,U_2[e]\,U_3[g]$$

UB$[r_{1,3}, r_{2,3}]$

$$-U_1[f]\,U_2[b]\,U_3[e] - 2\,\alpha\,U_1[f]\,U_2[b]\,U_3[e] + U_1[b]\,U_2[f]\,U_3[e] + 2\,\alpha\,U_1[b]\,U_2[f]\,U_3[e] + \frac{1}{2}\,\epsilon\,U_1[g]\,U_2[f]\,U_3[e] - \frac{1}{2}\,\epsilon\,U_1[f]\,U_2[g]\,U_3[e]$$