Cheat Sheet Quantum Groups

Every Formula Worth Dreading

[Ma1, Example 1.3.2]. A Hopf algebra over $\mathbb{Q}[q^{\pm 1}]$:

$$\langle 1, X, g^{\pm 1}\rangle \Big/ \left(Xg^{\pm 1} = q^{\pm 1} g^{\pm 1} X\right),$$

with $\Delta\colon (X,\ g^{\pm 1}) \mapsto (X\otimes 1 + g\otimes X,\ g^{\pm 1}\otimes g^{\pm 1})$, $\epsilon\colon (X,\ g^{\pm 1}) \mapsto (0,\ 1)$, and $S: (X,\ g^{\pm 1}) \mapsto (-g^{-1}X,\ g^{\mp 1})$. Has $S^2 \neq 1$.

[CP, Definition-Proposition 6.4.3], $U_h(sl_2)$ in detail:

DEFINITION–PROPOSITION 6.4.3 *Quantum $sl_2(\mathbb{C})$ is the topological Hopf algebra $U_h(sl_2(\mathbb{C}))$ over $\mathbb{C}[[h]]$ defined as follows.*

Let $P = \mathbb{C}\{H, X^+, X^-\}$ be the algebra of non-commutative polynomials in three generators H, X^+ and X^-, let I be the two-sided ideal of $P[[h]]$ generated by

$$[H, X^+] - 2X^+,\quad [H, X^-] + 2X^-,\quad [X^+, X^-] - \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \tag{27}$$

and let $\bar{I}$ be the closure of I in the h-adic topology. Then, $U_h(sl_2(\mathbb{C})) = P[[h]]/\bar{I}$ as an algebra over $\mathbb{C}[[h]]$.

There are homomorphisms of $\mathbb{C}[[h]]$-algebras

$$\Delta_h : U_h(sl_2(\mathbb{C})) \to U_h(sl_2(\mathbb{C})) \otimes U_h(sl_2(\mathbb{C})),$$

$$\epsilon_h : U_h(sl_2(\mathbb{C})) \to \mathbb{C}[[h]], \qquad S_h : U_h(sl_2(\mathbb{C})) \to U_h(sl_2(\mathbb{C})),$$

(an anti-homomorphism in the case S_h) given on generators by (19) and (21)–(24), which define the structure of a topological Hopf algebra on $U_h(sl_2(\mathbb{C}))$.

Moreover, $U_h(sl_2(\mathbb{C}))$ is a QUE algebra whose classical limit is the Lie bialgebra structure on $sl_2(\mathbb{C})$ defined by $r = X^+ \wedge X^-$.

$$\Delta_h(H) = H \otimes 1 + 1 \otimes H \tag{19}$$

$$\Delta_h(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+ \tag{21}$$

$$S_h(H) = -H,\quad S_h(X^+) = -X^+ e^{-hH},\quad \epsilon_h(H) = \epsilon_h(X^+) = h \tag{22}$$

$$\Delta_h(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^- \tag{23}$$

$$S_h(X^-) = -e^{hH} X^-,\quad \epsilon_h(X^-) = 0 \tag{24}$$

[CP, Proposition 6.4.8], R for $U_h(sl_2)$: With $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and $[n]_q! = [n]_q \cdots [1]_q$,

PROPOSITION 6.4.8 *The Hopf algebra $U_h(sl_2(\mathbb{C}))$ is topologically quasitriangular with universal R-matrix*

$$\mathcal{R}_h = \sum_{n=0}^{\infty} R_n(h) e^{\frac{1}{2}h(H\otimes H)} (X^+)^n \otimes (X^-)^n, \tag{30}$$

where

$$R_n(h) = \frac{q^{\frac{1}{2}n(n+1)}(1-q^{-2})^n}{[n]_q!}, \qquad (q = e^h).$$

[Ma2, Example 18.8], the QG structure on a general SS $\mathfrak{g}$ (seems to have no universal R formula):

Example 18.8 *Given a Cartan and root datum and $H, B, \tilde{B}$ as above, and $q \in k^*$ such that $q_i^2 \neq 1$ for all i, we have an ordinary Hopf algebra $U_q = B' \bowtie H \bowtie B^{\mathrm{op}}$. It consists of generators e_i, f_i, K_μ, the relations*

$$K_\mu K_\nu = K_{\mu+\nu},\quad K_\mu e^i = q^{\langle \mu, i'\rangle} e^i K_\mu,\quad f_i K_\mu = q^{\langle \mu, i'\rangle} K_\mu f_i,$$

$$[e^i, f_j] = \frac{K_i^{\frac{i\cdot i}{2}} - K_i^{-\frac{i\cdot i}{2}}}{q_i - q_i^{-1}} \delta^i_j$$

and the relations coming from the kernels of ev. *The coalgebra is*

$$\Delta e^i = e^i \otimes K_i^{\frac{i\cdot i}{2}} + 1 \otimes e^i,\quad \Delta f_i = f_i \otimes 1 + K_i^{-\frac{i\cdot i}{2}} \otimes f_i,$$

$$\epsilon(K_\mu) = 1,\quad \epsilon(e^i) = \epsilon(f_i) = 0.$$

The antipode has the usual form uniquely determined by the above.

[CP, Theorem 8.3.9], the universal R for a general SS $\mathfrak{g}$:

THEOREM 8.3.9 *For any finite-dimensional complex simple Lie algebra $\mathfrak{g}$, $U_h(\mathfrak{g})$ is topologically quasitriangular with universal R-matrix*

$$\mathcal{R}_h = \exp\left[h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j\right] \prod_\beta \exp_{q_\beta}[(1 - q_\beta^{-2}) X_\beta^+ \otimes X_\beta^-],$$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the β_r-term appears to the left of the β_s-term if $r > s$ (see 8.1.4). ∎

[Ma2, Section 8], the quantum double: $\mathcal{D}A := A^{*,op} \otimes A$ with $(\phi a)(\psi b) := \langle S a_1, \psi_1\rangle\langle a_3, \psi_3\rangle(\psi_2\phi)(a_2 b)$, $R = Id \in A^* \otimes A \subset \mathcal{D}A \otimes \mathcal{D}A$. What problem does it solve?

In QGverse, why start by deforming the co-product?

[ES, Section 3.3] For $(\mathfrak{g}, [,])$ and $r \in \Lambda^2(\mathfrak{g})$, $\delta a := [r, a\otimes 1 + 1 \otimes a]$ makes $\mathfrak{g}$ a "coboundary" Lie bialgebra iff $\mathrm{CYB}(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$ is $\mathfrak{g}$-invariant. "Triangular" means $\mathrm{CYB}(r) = 0$.

References.

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