Loseu @ Colloq: Orbit method via deformations of singular symplectic varieties

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primitive ideals.

$\mathfrak{g}$: f.d. Lie alg / $\mathbb{C}$

Ex $\mathfrak{g}=sl_2=\left\{\begin{pmatrix}a & b\\ c & -a\end{pmatrix}: a,b,c\in\mathbb{C}\right\}$ in blue

Q: Classify all irreps of $\mathfrak{g}$

too hard! Instead, classify annihilator ideals in $U(\mathfrak{g})$

$M\in\mathfrak{g}$-rep $\longrightarrow$ $Ann(M)=\{a\in U(\mathfrak{g})\mid a_M=0\}$ a 2-sided ideal.

Def A primitive ideal is the annihilator ideal of some rep.

$Prim(\mathfrak{g}):=\{\text{primitive ideals}\}$

$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $[h,e]=2e$ $[h,f]=-2f$ $[e,f]=h$

Prim $\mathfrak{g}$ has two series:

$n>0 \rightsquigarrow Ann(\mathbb{C}^n)$

$z\in\mathbb{C} \rightsquigarrow I_z=U(\mathfrak{g})(4fe+(h+1)^2-z)$

$\mathfrak{g}\rightsquigarrow$ Lie group $G\circlearrowright\mathfrak{g} \rightsquigarrow G\circlearrowright\mathfrak{g}^*$

$\mathfrak{g}^*/G=$ set of orbits

Thm (Dixmier, 1963)

If $\mathfrak{g}$ is nilpotent there is a natural bijection between

$$\mathfrak{g}^*/G \xrightarrow{\sim} \mathrm{Prim}(\mathfrak{g})$$

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orbit method for ss $\mathfrak{g}$ (sl_n, so_n, sp_n, exceptionals)

$$\mathfrak{g}^*/G \cong \mathfrak{g}/G$$

For sl_n, $\mathfrak{g}/G$ is described by Jordan normal forms.

Prim $\mathfrak{g}$ is complicated (1980's); not directly related to orbits and in some sense it is larger.

<u>Thm</u> (Loseu, 2011) $\exists$ nat'l map

$$\mathfrak{g}/G \longrightarrow \mathrm{Prim}(G)$$

"almost always" injective (conjecturally always)

$\{\mathcal{O}\} \subset sl_2 \longmapsto \mathrm{Ann}(\mathbb{C})$ $\quad \mathcal{O}_z = \{A \neq 0 : \det(A) = -z\} = G\begin{pmatrix} \sqrt{z} & 1 \\ 0 & -\sqrt{z} \end{pmatrix}$

$$\mathcal{O}_z \longmapsto I_z$$

<u>Singular sympl. varieties:</u>

A smooth sympl. variety is a smooth alg. variety X w/ alg $\mathbb{C}$-sympl. form ω.

example: $X = \mathbb{C}^{2n}$ $\quad \omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$

<u>Def</u> A sing. sympl. variety SSV (Beauville 2000) is a normal alg. variety X s.t.

1. X^{reg} is symplectic w/ form ω
2. $\exists$ res. of sing. $\tilde{X} \xrightarrow[\rho]{} X$ s.t. $\rho^*(\omega)$ extends from $\rho^{-1}(X^{reg})$ to all of $\tilde{X}$.

Ex. $X = \overline{\mathcal{O}_0} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \right\}$

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