

Pensieve header: The loose \$gl_n\$ (failed).

Two salvation parameters - ϵ and η .

$$h_{\text{here}} = h_{\text{naive}} - \epsilon g; h_{\text{naive}} = h_{\text{here}} + \epsilon g.$$

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 $\chi_{\text{cond}} := \text{If}[\text{TrueQ}@\text{cond}, 1, 0];$ 
 $\mathbf{B}[\mathbf{0}, \_ ] = \mathbf{0}; \mathbf{B}[\_, \mathbf{0}] = \mathbf{0};$ 
 $\mathbf{B}[\mathbf{c}_- * \mathbf{x} : (\mathbf{e} | \mathbf{f} | \mathbf{g} | \mathbf{h})\_\_, \mathbf{y}_-] := \text{Expand}[\mathbf{c} \mathbf{B}[\mathbf{x}, \mathbf{y}]];$ 
 $\mathbf{B}[\mathbf{y}_-, \mathbf{c}_- * \mathbf{x} : (\mathbf{e} | \mathbf{f} | \mathbf{g} | \mathbf{h})\_\_] := \text{Expand}[\mathbf{c} \mathbf{B}[\mathbf{y}, \mathbf{x}]];$ 
 $\mathbf{B}[\mathbf{x\_Plus}, \mathbf{y}_-] := \mathbf{B}[\#, \mathbf{y}] \& /@ \mathbf{x};$ 
 $\mathbf{B}[\mathbf{x}_-, \mathbf{y\_Plus}] := \mathbf{B}[\mathbf{x}, \#] \& /@ \mathbf{y};$ 

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 $\mathbf{B}[\mathbf{h}_-, \_ ] = \mathbf{0}; \mathbf{B}[\mathbf{g}_-, \mathbf{g}_-] = \mathbf{0};$ 

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 $\mathbf{B}[\mathbf{e}_{i,j}, \mathbf{e}_{k,l}] := \eta \chi_{j=k} \mathbf{e}_{i,l} - \eta \chi_{l=i} \mathbf{e}_{k,j};$ 
 $\mathbf{B}[\mathbf{f}_{i,j}, \mathbf{f}_{k,l}] := \eta \epsilon \chi_{j=k} \mathbf{f}_{i,l} - \eta \epsilon \chi_{l=i} \mathbf{f}_{k,j};$ 
 $\mathbf{B}[\mathbf{e}_{i,j}, \mathbf{f}_{k,l}] := \text{Expand}[\chi_{j=k} \eta (\epsilon \chi_{i<l} \mathbf{e}_{i,l} + \chi_{i>l} \mathbf{f}_{i,l}) -$ 
 $\chi_{l=i} \eta (\epsilon \chi_{k<j} \mathbf{e}_{k,j} + \chi_{k>j} \mathbf{f}_{k,j}) + \chi_{j=k \wedge i=l} \left( \frac{1}{2} (\mathbf{h}_i - \mathbf{h}_j) + \epsilon (\mathbf{g}_i - \mathbf{g}_j) \right)];$ 

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 $\mathbf{B}[\mathbf{g}_i, \mathbf{e}_{j,k}] := (\chi_{i=j} - \chi_{i=k}) \mathbf{e}_{j,k};$ 

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 $\mathbf{B}[\mathbf{g}_i, \mathbf{f}_{j,k}] := (\chi_{i=j} - \chi_{i=k}) \mathbf{f}_{j,k};$ 

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 $\mathbf{B}[\mathbf{y}_-, \mathbf{x}_-] := \text{Expand}[-\mathbf{B}[\mathbf{x}, \mathbf{y}]];$ 

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 $\text{Basis}^+[\mathbf{n}_-] := \text{Union}@\text{Flatten}@\{\text{Table}[\mathbf{e}_{i,j}, \{\mathbf{i}, \mathbf{n}-1\}, \{\mathbf{j}, \mathbf{i}+1, \mathbf{n}\}], \text{Table}[\mathbf{g}_i, \{\mathbf{i}, \mathbf{n}\}]\};$ 
 $\text{Basis}^-[\mathbf{n}_-] := \text{Union}@\text{Flatten}@\{\text{Table}[\mathbf{f}_{i,j}, \{\mathbf{i}, 2, \mathbf{n}\}, \{\mathbf{j}, \mathbf{i}-1\}], \text{Table}[\mathbf{h}_i, \{\mathbf{i}, \mathbf{n}\}]\};$ 
 $\text{Basis}[\mathbf{n}_-] := \text{Union}[\text{Basis}^+[\mathbf{n}], \text{Basis}^-[\mathbf{n}]];$ 

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Basis[4]

{g₁, g₂, g₃, g₄, h₁, h₂, h₃, h₄, e_{1,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{2,4}, e_{3,4}, f_{2,1}, f_{3,1}, f_{3,2}, f_{4,1}, f_{4,2}, f_{4,3}}

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n = 3;
Table[
  {x, y} → B[x, y],
  {x, Basis[n]}, {y, Basis[n]}
] // MatrixForm

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$\{g_1, g_1\} \rightarrow 0$	$\{g_1, g_2\} \rightarrow 0$	$\{g_1, g_3\} \rightarrow 0$	$\{g_1, h_1\} \rightarrow 0$	$\{g_1, h_2\} \rightarrow 0$	$\{g_1, h_3\} \rightarrow 0$
$\{g_2, g_1\} \rightarrow 0$	$\{g_2, g_2\} \rightarrow 0$	$\{g_2, g_3\} \rightarrow 0$	$\{g_2, h_1\} \rightarrow 0$	$\{g_2, h_2\} \rightarrow 0$	$\{g_2, h_3\} \rightarrow 0$
$\{g_3, g_1\} \rightarrow 0$	$\{g_3, g_2\} \rightarrow 0$	$\{g_3, g_3\} \rightarrow 0$	$\{g_3, h_1\} \rightarrow 0$	$\{g_3, h_2\} \rightarrow 0$	$\{g_3, h_3\} \rightarrow 0$
$\{h_1, g_1\} \rightarrow 0$	$\{h_1, g_2\} \rightarrow 0$	$\{h_1, g_3\} \rightarrow 0$	$\{h_1, h_1\} \rightarrow 0$	$\{h_1, h_2\} \rightarrow 0$	$\{h_1, h_3\} \rightarrow 0$
$\{h_2, g_1\} \rightarrow 0$	$\{h_2, g_2\} \rightarrow 0$	$\{h_2, g_3\} \rightarrow 0$	$\{h_2, h_1\} \rightarrow 0$	$\{h_2, h_2\} \rightarrow 0$	$\{h_2, h_3\} \rightarrow 0$
$\{h_3, g_1\} \rightarrow 0$	$\{h_3, g_2\} \rightarrow 0$	$\{h_3, g_3\} \rightarrow 0$	$\{h_3, h_1\} \rightarrow 0$	$\{h_3, h_2\} \rightarrow 0$	$\{h_3, h_3\} \rightarrow 0$
$\{e_{1,2}, g_1\} \rightarrow -e_{1,2}$	$\{e_{1,2}, g_2\} \rightarrow e_{1,2}$	$\{e_{1,2}, g_3\} \rightarrow 0$	$\{e_{1,2}, h_1\} \rightarrow 0$	$\{e_{1,2}, h_2\} \rightarrow 0$	$\{e_{1,2}, h_3\} \rightarrow 0$
$\{e_{1,3}, g_1\} \rightarrow -e_{1,3}$	$\{e_{1,3}, g_2\} \rightarrow 0$	$\{e_{1,3}, g_3\} \rightarrow e_{1,3}$	$\{e_{1,3}, h_1\} \rightarrow 0$	$\{e_{1,3}, h_2\} \rightarrow 0$	$\{e_{1,3}, h_3\} \rightarrow 0$
$\{e_{2,3}, g_1\} \rightarrow 0$	$\{e_{2,3}, g_2\} \rightarrow -e_{2,3}$	$\{e_{2,3}, g_3\} \rightarrow e_{2,3}$	$\{e_{2,3}, h_1\} \rightarrow 0$	$\{e_{2,3}, h_2\} \rightarrow 0$	$\{e_{2,3}, h_3\} \rightarrow 0$
$\{f_{2,1}, g_1\} \rightarrow f_{2,1}$	$\{f_{2,1}, g_2\} \rightarrow -f_{2,1}$	$\{f_{2,1}, g_3\} \rightarrow 0$	$\{f_{2,1}, h_1\} \rightarrow 0$	$\{f_{2,1}, h_2\} \rightarrow 0$	$\{f_{2,1}, h_3\} \rightarrow 0$
$\{f_{3,1}, g_1\} \rightarrow f_{3,1}$	$\{f_{3,1}, g_2\} \rightarrow 0$	$\{f_{3,1}, g_3\} \rightarrow -f_{3,1}$	$\{f_{3,1}, h_1\} \rightarrow 0$	$\{f_{3,1}, h_2\} \rightarrow 0$	$\{f_{3,1}, h_3\} \rightarrow 0$
$\{f_{3,2}, g_1\} \rightarrow 0$	$\{f_{3,2}, g_2\} \rightarrow f_{3,2}$	$\{f_{3,2}, g_3\} \rightarrow -f_{3,2}$	$\{f_{3,2}, h_1\} \rightarrow 0$	$\{f_{3,2}, h_2\} \rightarrow 0$	$\{f_{3,2}, h_3\} \rightarrow 0$

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n = 4;
Union@Table[
  {x, y} = t; B[x, y] + B[y, x],
  {t, Tuples[Basis[n], 2]}
]
{0}

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$n = 3$;

DeleteCases [Table [

{ $\{x, y, z\} = t\} \rightarrow B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]]$,

{ t , Tuples[Basis[n], 3]}

], _ $\rightarrow \emptyset$]

{ $\{e_{1,2}, e_{1,3}, f_{2,1}\} \rightarrow \epsilon e_{1,3} - \epsilon \eta^2 e_{1,3}$, $\{e_{1,2}, e_{1,3}, f_{3,1}\} \rightarrow -\epsilon e_{1,2} + \epsilon \eta^2 e_{1,2}$,
 $\{e_{1,2}, e_{2,3}, f_{2,1}\} \rightarrow -\epsilon e_{2,3} + \epsilon \eta^2 e_{2,3}$, $\{e_{1,2}, e_{2,3}, f_{3,2}\} \rightarrow \epsilon e_{1,2} - \epsilon \eta^2 e_{1,2}$,
 $\{e_{1,2}, f_{2,1}, e_{1,3}\} \rightarrow -\epsilon e_{1,3} + \epsilon \eta^2 e_{1,3}$, $\{e_{1,2}, f_{2,1}, e_{2,3}\} \rightarrow \epsilon e_{2,3} - \epsilon \eta^2 e_{2,3}$,
 $\{e_{1,2}, f_{2,1}, f_{3,1}\} \rightarrow \epsilon f_{3,1} - \epsilon \eta^2 f_{3,1}$, $\{e_{1,2}, f_{2,1}, f_{3,2}\} \rightarrow -\epsilon f_{3,2} + \epsilon \eta^2 f_{3,2}$,
 $\{e_{1,2}, f_{3,1}, e_{1,3}\} \rightarrow \epsilon e_{1,2} - \epsilon \eta^2 e_{1,2}$, $\{e_{1,2}, f_{3,1}, f_{2,1}\} \rightarrow -\epsilon f_{3,1} + \epsilon \eta^2 f_{3,1}$,
 $\{e_{1,2}, f_{3,2}, e_{2,3}\} \rightarrow -\epsilon e_{1,2} + \epsilon \eta^2 e_{1,2}$, $\{e_{1,2}, f_{3,2}, f_{2,1}\} \rightarrow \epsilon f_{3,2} - \epsilon \eta^2 f_{3,2}$,
 $\{e_{1,3}, e_{1,2}, f_{2,1}\} \rightarrow -\epsilon e_{1,3} + \epsilon \eta^2 e_{1,3}$, $\{e_{1,3}, e_{1,2}, f_{3,1}\} \rightarrow \epsilon e_{1,2} - \epsilon \eta^2 e_{1,2}$,
 $\{e_{1,3}, e_{2,3}, f_{3,1}\} \rightarrow \epsilon e_{2,3} - \epsilon \eta^2 e_{2,3}$, $\{e_{1,3}, e_{2,3}, f_{3,2}\} \rightarrow -\epsilon e_{1,3} + \epsilon \eta^2 e_{1,3}$,
 $\{e_{1,3}, f_{2,1}, e_{1,2}\} \rightarrow \epsilon e_{1,3} - \epsilon \eta^2 e_{1,3}$, $\{e_{1,3}, f_{2,1}, f_{3,1}\} \rightarrow -\epsilon f_{2,1} + \epsilon \eta^2 f_{2,1}$,
 $\{e_{1,3}, f_{3,1}, e_{1,2}\} \rightarrow -\epsilon e_{1,2} + \epsilon \eta^2 e_{1,2}$, $\{e_{1,3}, f_{3,1}, e_{2,3}\} \rightarrow -\epsilon e_{2,3} + \epsilon \eta^2 e_{2,3}$,
 $\{e_{1,3}, f_{3,1}, f_{2,1}\} \rightarrow \epsilon f_{2,1} - \epsilon \eta^2 f_{2,1}$, $\{e_{1,3}, f_{3,1}, f_{3,2}\} \rightarrow \epsilon f_{3,2} - \epsilon \eta^2 f_{3,2}$,
 $\{e_{1,3}, f_{3,2}, e_{2,3}\} \rightarrow \epsilon e_{1,3} - \epsilon \eta^2 e_{1,3}$, $\{e_{1,3}, f_{3,2}, f_{3,1}\} \rightarrow -\epsilon f_{3,2} + \epsilon \eta^2 f_{3,2}$,
 $\{e_{2,3}, e_{1,2}, f_{2,1}\} \rightarrow \epsilon e_{2,3} - \epsilon \eta^2 e_{2,3}$, $\{e_{2,3}, e_{1,2}, f_{3,2}\} \rightarrow -\epsilon e_{1,2} + \epsilon \eta^2 e_{1,2}$,
 $\{e_{2,3}, e_{1,3}, f_{3,1}\} \rightarrow -\epsilon e_{2,3} + \epsilon \eta^2 e_{2,3}$, $\{e_{2,3}, e_{1,3}, f_{3,2}\} \rightarrow \epsilon e_{1,3} - \epsilon \eta^2 e_{1,3}$,
 $\{e_{2,3}, f_{2,1}, e_{1,2}\} \rightarrow -\epsilon e_{2,3} + \epsilon \eta^2 e_{2,3}$, $\{e_{2,3}, f_{2,1}, f_{3,2}\} \rightarrow \epsilon f_{2,1} - \epsilon \eta^2 f_{2,1}$,
 $\{e_{2,3}, f_{3,1}, e_{1,3}\} \rightarrow \epsilon e_{2,3} - \epsilon \eta^2 e_{2,3}$, $\{e_{2,3}, f_{3,1}, f_{3,2}\} \rightarrow -\epsilon f_{3,1} + \epsilon \eta^2 f_{3,1}$,
 $\{e_{2,3}, f_{3,2}, e_{1,2}\} \rightarrow \epsilon e_{1,2} - \epsilon \eta^2 e_{1,2}$, $\{e_{2,3}, f_{3,2}, e_{1,3}\} \rightarrow -\epsilon e_{1,3} + \epsilon \eta^2 e_{1,3}$,
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 $\{f_{3,2}, f_{3,1}, e_{1,3}\} \rightarrow -\epsilon f_{3,2} + \epsilon \eta^2 f_{3,2}$, $\{f_{3,2}, f_{3,1}, e_{2,3}\} \rightarrow \epsilon f_{3,1} - \epsilon \eta^2 f_{3,1}$ }