Pensieve header: Implementing and verifying gl_n^ϵ; with highlighted centre. Continues pensieve://2017-01/, continued pensieve://2020-01/.

$h_{\text{here}} = h_{\text{naive}} - \epsilon g$; $h_{\text{naive}} = h_{\text{here}} + \epsilon g$.

```
χ_cond_ := If[TrueQ@cond, 1, 0];
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : (e | f | g | h)__, y_] := Expand[c B[x, y]];
B[y_, c_ * x : (e | f | g | h)__] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
P[0, _] = 0; P[_, 0] = 0;
P[c_ * x : (e | f | g | h)__, y_] := Expand[c P[x, y]];
P[y_, c_ * x : (e | f | g | h)__] := Expand[c P[y, x]];
P[x_Plus, y_] := P[#, y] & /@ x;
P[x_, y_Plus] := P[x, #] & /@ y;
```

```
P[e_{i_,j_}, f_{k_,l_}] := χ_{j==k∧i==l}; P[f_{k_,l_}, e_{i_,j_}] := χ_{j==k∧i==l};
P[(e | f)__, h_] = 0; P[h_, (e | f)__] = 0;
P[g_{i_}, h_{j_}] := 2 χ_{i==j}; P[h_{j_}, g_{i_}] := 2 χ_{i==j}; P[h_{i_}, h_{j_}] := -4 ϵ χ_{i==j};
P[(e | g)__, (e | g)__] = 0;
P[(f | g)__, (f | g)__] = 0;
```

```
B[h_, _] = 0; B[g_, g_] = 0;

B[e_{i_,j_}, e_{k_,l_}] := χ_{j==k} e_{i,l} - χ_{l==i} e_{k,j};
B[f_{i_,j_}, f_{k_,l_}] := ϵ χ_{j==k} f_{i,l} - ϵ χ_{l==i} f_{k,j};
B[e_{i_,j_}, f_{k_,l_}] := Expand[χ_{j==k} (ϵ χ_{i<l} e_{i,l} + χ_{i>l} f_{i,l}) -
    χ_{l==i} (ϵ χ_{k<j} e_{k,j} + χ_{k>j} f_{k,j}) + χ_{j==k∧i==l} (1/2 (h_i - h_j) + ϵ (g_i - g_j))];

B[g_{i_}, e_{j_,k_}] := (χ_{i==j} - χ_{i==k}) e_{j,k};
B[g_{i_}, f_{j_,k_}] := (χ_{i==j} - χ_{i==k}) f_{j,k};
```

```
B[y_, x_] := Expand[-B[x, y]];
```

```
Basis⁺[n_] := Union@Flatten@{Table[e_{i,j}, {i, n - 1}, {j, i + 1, n}], Table[g_i, {i, n}]};
Basis⁻[n_] := Union@Flatten@{Table[f_{i,j}, {i, 2, n}, {j, i - 1}], Table[h_i, {i, n}]};
Basis[n_] := Union[Basis⁺[n], Basis⁻[n]];
```

```
Basis[4]
```

$$\{g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4, e_{1,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{2,4}, e_{3,4}, f_{2,1}, f_{3,1}, f_{3,2}, f_{4,1}, f_{4,2}, f_{4,3}\}$$

```
n = 3;
Table[
  {x, y} → B[x, y],
  {x, Basis[n]}, {y, Basis[n]}
 ] // MatrixForm
```

$$\begin{pmatrix}
\{g_1, g_1\} \to 0 & \{g_1, g_2\} \to 0 & \{g_1, g_3\} \to 0 & \{g_1, e_{1,2}\} \to e_{1,2} \\
\{g_2, g_1\} \to 0 & \{g_2, g_2\} \to 0 & \{g_2, g_3\} \to 0 & \{g_2, e_{1,2}\} \to -e_{1,2} \\
\{g_3, g_1\} \to 0 & \{g_3, g_2\} \to 0 & \{g_3, g_3\} \to 0 & \{g_3, e_{1,2}\} \to 0 \\
\{e_{1,2}, g_1\} \to -e_{1,2} & \{e_{1,2}, g_2\} \to e_{1,2} & \{e_{1,2}, g_3\} \to 0 & \{e_{1,2}, e_{1,2}\} \to 0 \\
\{e_{1,3}, g_1\} \to -e_{1,3} & \{e_{1,3}, g_2\} \to 0 & \{e_{1,3}, g_3\} \to e_{1,3} & \{e_{1,3}, e_{1,2}\} \to 0 \\
\{e_{2,3}, g_1\} \to 0 & \{e_{2,3}, g_2\} \to -e_{2,3} & \{e_{2,3}, g_3\} \to e_{2,3} & \{e_{2,3}, e_{1,2}\} \to -e_{1,3} \\
\{h_1, g_1\} \to 0 & \{h_1, g_2\} \to 0 & \{h_1, g_3\} \to 0 & \{h_1, e_{1,2}\} \to 0 \\
\{h_2, g_1\} \to 0 & \{h_2, g_2\} \to 0 & \{h_2, g_3\} \to 0 & \{h_2, e_{1,2}\} \to 0 \\
\{h_3, g_1\} \to 0 & \{h_3, g_2\} \to 0 & \{h_3, g_3\} \to 0 & \{h_3, e_{1,2}\} \to 0 \\
\{f_{2,1}, g_1\} \to f_{2,1} & \{f_{2,1}, g_2\} \to -f_{2,1} & \{f_{2,1}, g_3\} \to 0 & \{f_{2,1}, e_{1,2}\} \to -\in g_1 + \in g_2 - \frac{h_1}{2} + \frac{h_2}{2} \\
\{f_{3,1}, g_1\} \to f_{3,1} & \{f_{3,1}, g_2\} \to 0 & \{f_{3,1}, g_3\} \to -f_{3,1} & \{f_{3,1}, e_{1,2}\} \to f_{3,2} \\
\{f_{3,2}, g_1\} \to 0 & \{f_{3,2}, g_2\} \to f_{3,2} & \{f_{3,2}, g_3\} \to -f_{3,2} & \{f_{3,2}, e_{1,2}\} \to 0
\end{pmatrix}$$

```
n = 4;
Union@Table[
  {x, y} = t; B[x, y] + B[y, x],
  {t, Tuples[Basis[n], 2]}
 ]
```

```
{0}
```

```
n = 3;
DeleteCases[Flatten@Table[
   {x, y, z} → P[B[x, y], z] + P[y, B[x, z]],
   {x, Basis[n]}, {y, Basis[n]}, {z, Basis[n]}
  ], _ → 0]
```

```
{}
```

```
n = 3;
DeleteCases[Table[
  ({x, y, z} = t) → B[x, B[y, z]] + B[y, B[z, x]] + B[z, B[x, y]],
  {t, Tuples[Basis[n], 3]}
 ], _ → 0]
```

```
{}
```

```
$TD = ∞; ħ /: ħ^d_· /; d > $TD := 0;
```

```
PBWRule = {};
x_ ≼ y_ := OrderedQ[{x, y} /. PBWRule]; x_ ≺ y_ := ! OrderedQ[{y, x} /. PBWRule];
UU_i_[ℰ_] := ℰ /. (x : e | f | g | h)_s__ :↦ U_i[x_s];
Simp[ℰ_] := Collect[ℰ, Times[U_[__] ..], Expand];
Simp[ℰ_] := Expand[ℰ];
```

```
a U1[a, b] U2[c, d] + b U1[a, b] U2[c, d] + c U1[a, a] U2[c, d] + a U1[a, c] + b U1[a, c] // Simp
```

$$(a+b)\,U_1[a,c] + c\,U_1[a,a]\,U_2[c,d] + (a+b)\,U_1[a,b]\,U_2[c,d]$$

```
-∈ g1 + ∈ g2 - h1/2 + h2/2 + 2 e1,2 // UU3
```

$$-\in U_3[g_1] + \in U_3[g_2] - \frac{1}{2}U_3[h_1] + \frac{1}{2}U_3[h_2] + 2\,U_3[e_{1,2}]$$

```
m_s__[0] = 0;
m_s__[x_Plus] := m_s /@ x;
```

```
m_i_→j_[ℰ_] := ℰ /. U_i → U_j;
```

```
m1→4[U1[] U2[g2, g3] U3[e2,3]]
```

$$U_2[g_2, g_3]\,U_3[e_{2,3}]\,U_4[]$$

```
m_i_,j_→k_[c_. U_i_[x___] U_j_[]] := c U_k[x];
m_i_,j_→k_[c_. U_i_[] U_j_[y___]] := c U_k[y];
m_i_,j_→k_[c_. U_i_[xx___, x_] U_j_[y_, yy___]] := If[x ≼ y,
   c U_k[xx, x, y, yy],
   ((U_i[xx] (U_j[y, x] + UU_j[B[x, y]]) // Expand // m_i,j→i) U_j[yy] // Expand // m_i,j→k)
     c // Simp
  ];
```

```
c U1[a, a] U2[c, d] + (a + b) U1[a, b] U2[c, d] // m1,2→3
```

$$c\,U_3[a,a,c,d] + (a+b)\,U_3[a,b,c,d]$$

```
UProducts[n_, {}, 0] = {1}; UProducts[n_, {}, d_Integer] /; d > 0 = {};
UProducts[n_, {i_, is___}, d_Integer] := Sort@Flatten@
    Table[(U_i @@@ Subsets[Basis[n], {j}]) u, {j, 0, d}, {u, UProducts[n, {is}, d - j]}];
```

```
UProducts[1, {1}, 1]
```

$$\{U_1[g_1], U_1[h_1]\}$$

```
Union[(u ⟼ m1,3→1[m1,2→1[u]] - m1,2→1[m2,3→2[u]]) /@ UProducts[3, {1, 2, 3}, 3]]
```

$$\{0\}$$

```
S[ℰ_] := Union@Cases[{ℰ}, U_i_[___] :↦ i, ∞];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
x_ ** y_ := Module[{is = S[x] ⋂ S[y], σ, z},
  z = x; Do[z = m_{i→σ@i}[z], {i, is}];
  z = Expand[y z]; Do[z = m_{σ@i,i→i}[z], {i, is}]; z]
```

```
{x = RandomChoice[UProducts[3, {1, 2}, 3]],
 y = RandomChoice[UProducts[3, {1, 2, 3}, 3]], x ** y}
```

$$\{U_1[h_2]\,U_2[g_3, f_{3,2}],\ U_1[h_2, f_{3,2}]\,U_2[f_{2,1}]\,U_3[],$$
$$\in U_1[h_2, h_2, f_{3,2}]\,U_2[g_3, f_{3,1}]\,U_3[] + U_1[h_2, h_2, f_{3,2}]\,U_2[g_3, f_{2,1}, f_{3,2}]\,U_3[]\}$$