

Pensieve header: Conjugating the exponentials of upper-triangular matrices to polynomial form.

`MatrixExp[$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$] // MatrixForm`

$$\begin{pmatrix} e^a & \frac{b e^{-a} (-1 + e^{2a})}{2a} \\ 0 & e^{-a} \end{pmatrix}$$

`n = 3; M = Table[If[i ≤ j, ai,j, 0], {i, n}, {j, n}];`

`M // MatrixForm`

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix}$$

`MatrixExp[M] // Simplify // MatrixForm`

$$\begin{pmatrix} e^{a_{1,1}} & \frac{(e^{a_{1,1}} - e^{a_{2,2}}) a_{1,2}}{a_{1,1} - a_{2,2}} - \frac{e^{a_{3,3}} (a_{1,1} - a_{2,2}) (-a_{1,2} a_{2,3} + a_{1,3} (a_{2,2} - a_{3,3})) - e^{a_{2,2}} a_{1,2} a_{2,3} (a_{1,1} - a_{3,3}) + e^{a_{1,1}} (a_{1,1} a_{1,3} - a_{1,3} a_{2,2} + a_{1,2} a_{2,3}) (a_{2,2} - a_{3,3})}{(a_{1,1} - a_{2,2}) (a_{1,1} - a_{3,3}) (a_{2,2} - a_{3,3})} & 0 \\ 0 & e^{a_{2,2}} & \frac{(e^{a_{2,2}} - e^{a_{3,3}}) a_{2,3}}{a_{2,2} - a_{3,3}} \\ 0 & 0 & e^{a_{3,3}} \end{pmatrix}$$

$$P = \begin{pmatrix} (a_{1,1} - a_{2,2}) & (a_{1,1} - a_{3,3}) & (a_{2,2} - a_{3,3}) & 0 & 0 \\ 0 & 0 & 0 & (a_{2,2} - a_{3,3}) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

`P.MatrixExp[M].Inverse[P] // Simplify // MatrixForm`

$$\begin{pmatrix} e^{a_{1,1}} & (e^{a_{1,1}} - e^{a_{2,2}}) a_{1,2} (a_{1,1} - a_{3,3}) - e^{a_{3,3}} (a_{1,1} - a_{2,2}) (-a_{1,2} a_{2,3} + a_{1,3} (a_{2,2} - a_{3,3})) - e^{a_{2,2}} a_{1,2} a_{2,3} (a_{1,1} - a_{3,3}) & 0 \\ 0 & e^{a_{2,2}} & 0 \\ 0 & 0 & e^{a_{3,3}} \end{pmatrix}$$

`n = 3;`

`M = Table[If[i ≤ j, ai,j, 0], {i, n}, {j, n}];`

`P = DiagonalMatrix[Table[$\prod_{i=k}^{n-1} \prod_{j=i+1}^n (a_{i,i} - a_{j,j})$, {k, n}]];`

`MatrixForm /@ {M, P}`

$$\left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix}, \begin{pmatrix} (a_{1,1} - a_{2,2}) & (a_{1,1} - a_{3,3}) & (a_{2,2} - a_{3,3}) & 0 & 0 \\ 0 & 0 & 0 & a_{2,2} - a_{3,3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

`P.MatrixExp[M].Inverse[P] // Simplify // MatrixForm`

$$\begin{pmatrix} e^{a_{1,1}} & (e^{a_{1,1}} - e^{a_{2,2}}) a_{1,2} (a_{1,1} - a_{3,3}) - e^{a_{3,3}} (a_{1,1} - a_{2,2}) (-a_{1,2} a_{2,3} + a_{1,3} (a_{2,2} - a_{3,3})) - e^{a_{2,2}} a_{1,2} a_{2,3} (a_{1,1} - a_{3,3}) & 0 \\ 0 & e^{a_{2,2}} & 0 \\ 0 & 0 & e^{a_{3,3}} \end{pmatrix}$$

```

n = 4;
M = Table[If[i ≤ j, ai,j, 0], {i, n}, {j, n}];
P = DiagonalMatrix[Table[ $\prod_{i=k}^{n-1} \prod_{j=i+1}^n (a_{i,i} - a_{j,j})$ ], {k, n}]];

```

```
MatrixForm /@ {M, P}
```

$$\left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}, \begin{pmatrix} (a_{1,1} - a_{2,2}) & (a_{1,1} - a_{3,3}) & (a_{2,2} - a_{3,3}) & (a_{1,1} - a_{4,4}) & (a_{2,2} - a_{4,4}) & (a_{3,3} - a_{4,4}) \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} \right.$$

```
P.MatrixExp[M].Inverse[P] // Simplify // MatrixForm
```

$$\begin{pmatrix} e^{a_{1,1}} & (e^{a_{1,1}} - e^{a_{2,2}}) a_{1,2} & (a_{1,1} - a_{3,3}) & (a_{1,1} - a_{4,4}) & (a_{1,1} (- (e^{a_{2,2}} - e^{a_{3,3}}) a_{1,2} a_{2,3} + (e^{a_{1,1}} - e^{a_{3,3}}) a_{1,3} (a_{2,2} \\ 0 & & e^{a_{2,2}} & & \\ 0 & & 0 & & \\ 0 & & 0 & & \end{pmatrix}$$

```
n = 3; ei,j := Normal[SparseArray[{i, j} → 1, {n, n}]]
```

```
e1,2 // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
Table[MatrixForm /@ (ei,j → MatrixExp[α ei,j]), {i, n}, {j, n}] // MatrixForm
```

$$\left(\begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^\alpha \end{pmatrix} \end{matrix} \right)$$

```
MatrixExp[α e1,2].MatrixExp[β e2,3] // MatrixForm
```

$$\begin{pmatrix} 1 & \alpha & \alpha\beta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixExp[β e2,3].MatrixExp[α e1,2] // MatrixForm
```

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixExp[β e2,3].MatrixExp[α e1,2].MatrixExp[α β e1,3] // MatrixForm
```

$$\begin{pmatrix} 1 & \alpha & \alpha\beta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

```
n = 3;
M = Table[Which[
  i > j, 0,
  i == j, a_i,
  i < j, (a_i - a_j) a_{i,j}
], {i, n}, {j, n}];
```

```
M // MatrixForm
```

$$\begin{pmatrix} a_1 & (a_1 - a_2) a_{1,2} & (a_1 - a_3) a_{1,3} \\ 0 & a_2 & (a_2 - a_3) a_{2,3} \\ 0 & 0 & a_3 \end{pmatrix}$$

```
MatrixExp[M] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} e^{a_1} (e^{a_1 - a_2}) a_{1,2} \frac{(e^{a_1 - a_3}) a_2 a_{1,2} a_{2,3} + a_3 ((-e^{a_1} + e^{a_3}) a_{1,3} - (e^{a_1 - a_2}) a_{1,2} a_{2,3}) + a_1 ((e^{a_1 - a_3}) a_{1,3} - (e^{a_2 - a_3}) a_{1,2} a_{2,3})}{a_1 - a_3} & & \\ 0 & e^{a_2} & (e^{a_2 - a_3}) a_{2,3} \\ 0 & 0 & e^{a_3} \end{pmatrix}$$

$$M = \begin{pmatrix} a_1 & (a_1 - a_3) (a_1 - a_2) a_{1,2} & (a_1 - a_3) a_{1,3} \\ 0 & a_2 & (a_1 - a_3) (a_2 - a_3) a_{2,3} \\ 0 & 0 & a_3 \end{pmatrix};$$

```
MatrixExp[M] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} e^{a_1} (e^{a_1 - a_2}) (a_1 - a_3) a_{1,2} (e^{a_1 - a_3}) a_{1,3} - (a_1 - a_3) ((e^{a_2 - a_3}) a_1 + (-e^{a_1} + e^{a_3}) a_2 + (e^{a_1 - a_2}) a_3) & & \\ 0 & e^{a_2} & (e^{a_2 - a_3}) (a_1 - a_3) a_{2,3} \\ 0 & 0 & e^{a_3} \end{pmatrix}$$

```
n = 4;
M = Table[Which[
  i > j, 0,
  i == j, a_i,
  i < j, (a_i - a_j) a_{i,j}
], {i, n}, {j, n}];
```

```
M // MatrixForm
```

$$\begin{pmatrix} a_1 & (a_1 - a_2) a_{1,2} & (a_1 - a_3) a_{1,3} & (a_1 - a_4) a_{1,4} \\ 0 & a_2 & (a_2 - a_3) a_{2,3} & (a_2 - a_4) a_{2,4} \\ 0 & 0 & a_3 & (a_3 - a_4) a_{3,4} \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

```
MatrixExp[M] // FullSimplify // MatrixForm
```

$$\begin{pmatrix} e^{a_1} (e^{a_1 - a_2}) a_{1,2} \frac{(e^{a_1 - a_3}) a_2 a_{1,2} a_{2,3} + a_3 ((-e^{a_1} + e^{a_3}) a_{1,3} - (e^{a_1 - a_2}) a_{1,2} a_{2,3}) + a_1 ((e^{a_1 - a_3}) a_{1,3} - (e^{a_2 - a_3}) a_{1,2} a_{2,3})}{a_1 - a_3} - e^{a_3} ((-a_1} & & \\ 0 & e^{a_2} & (e^{a_2 - a_3}) a_{2,3} \\ 0 & 0 & e^{a_3} \\ 0 & 0 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} a_1 & (a_1 - a_4) & (a_1 - a_3) & (a_1 - a_2) & a_{1,2} & (a_1 - a_4) & (a_1 - a_3) & a_{1,3} & (a_1 - a_4) & a_{1,4} \\ 0 & & a_2 & & & (a_2 - a_4) & (a_1 - a_3) & (a_2 - a_3) & a_{2,3} & (a_1 - a_4) & (a_2 - a_4) & a_{2,4} \\ 0 & & 0 & & & & a_3 & & & (a_1 - a_4) & (a_2 - a_4) & (a_3 - a_4) & a_4 \\ 0 & & 0 & & & & 0 & & & & & & a_4 \end{pmatrix}$$

MatrixExp[M] // FullSimplify // MatrixForm

$$\begin{pmatrix} e^{a_1} & (e^{a_1} - e^{a_2}) & (a_1 - a_3) & (a_1 - a_4) & a_{1,2} & -e^{a_2} & (a_1 - a_3)^2 & (a_1 - a_4) & (a_2 - a_4) & a_{1,2} & a_{2,3} & + e^{a_1} & (a_1 - a_4) & (a_{1,3} + \\ 0 & & e^{a_2} & & & & & & & & & & & \\ 0 & & 0 & & & & & & & & & & & \\ 0 & & 0 & & & & & & & & & & & \end{pmatrix}$$

```
n = 3;
M = Table[Which[
  i > j, 0,
  i == j, a_i,
  i < j, a_{i,j}
], {i, n}, {j, n}];
M // MatrixForm
```

$$\begin{pmatrix} a_1 & a_{1,2} & a_{1,3} \\ 0 & a_2 & a_{2,3} \\ 0 & 0 & a_3 \end{pmatrix}$$

$$M = \begin{pmatrix} a_1 & (a_1 - a_3) & a_{1,2} & (a_1 - a_3) & a_{1,3} \\ 0 & 0 & & (a_1 - a_3) & a_{2,3} \\ 0 & 0 & & a_3 & \end{pmatrix};$$

MatrixExp[M] // FullSimplify // MatrixForm

$$\begin{pmatrix} e^{a_1} & \frac{(-1 + e^{a_1}) (a_1 - a_3) a_{1,2}}{a_1} & \frac{-(-1 + e^{a_3}) a_1^2 a_{1,2} a_{2,3} - (-1 + e^{a_1}) a_1^2 a_{1,2} a_{2,3} + a_1 a_3 ((e^{a_1} - e^{a_3}) a_{1,3} + (-2 + e^{a_1} + e^{a_3}) a_{1,2} a_{2,3})}{a_1 a_3} \\ 0 & 1 & \frac{(-1 + e^{a_3}) (a_1 - a_3) a_{2,3}}{a_3} \\ 0 & 0 & e^{a_3} \end{pmatrix}$$