Pensieve header: Verifying the Duflo commutation relations, as communicated by Dylan in 1999.

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Date: Sun, 23 May 1999 17:24:53 +0200 (MET DST)
From: Dylan Thurston <Dylan.Thurston@math.unige.ch>
To: Dror Bar-Natan <drorbn@math.huji.ac.il>
Subject: Oh, yeah...

Duflo had another possibly useful comment: he said that there is a simpler
(non-trivial) algebra than sl_2, given by the relations (more or less)

        [x,y] = z
        [x,t] = x
        [y,t] = -y
        [t,z] = [x,z] = [y,z] = 0

I hope I got that right; it should be the Lie algebra of matrices

        0 x z
        0 t y
        0 0 0

He says that one can write down the exponential explicitly, that
commutators very quickly go to zero, etc.  Apparently it plays the same
role for reducible lie algebras that sl_2 plays for semi-simple ones.
It's worth thinking about.

Best,
        Dylan
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$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

{x.y - y.x == z, x.t - t.x == x, y.t - t.y == -y, t.z - z.t == o, x.z - z.x == o, y.z - z.y == o}

{True, True, True, True, True, True}

MatrixExp[α x + β y + γ z + δ t] // MatrixForm

$$\begin{pmatrix} 1 & \frac{(-1+e^{\delta})\,\alpha}{\delta} & \frac{-\alpha\,\beta+e^{\delta}\,\alpha\,\beta-\alpha\,\beta\,\delta+\gamma\,\delta^2}{\delta^2} \\ 0 & e^{\delta} & \frac{(-1+e^{\delta})\,\beta}{\delta} \\ 0 & 0 & 1 \end{pmatrix}$$

MatrixExp[α x].MatrixExp[β y].MatrixExp[γ z].MatrixExp[δ t] // MatrixForm

$$\begin{pmatrix} 1 & e^{\delta}\,\alpha & \alpha\,\beta+\gamma \\ 0 & e^{\delta} & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

Adding an “Euler” element:

```
d = (0 0 0; 0 0 0; 0 0 1); {d.x - x.d == o, d.t - t.d == o, d.y - y.d == -y, d.z - z.d == -z}
```

{True, True, True, True}

```
MatrixForm /@ {M = δ α x + δ β y + γ z + δ t, MatrixExp[M]}
```

$$\left\{\begin{pmatrix} 0 & \alpha\delta & \gamma \\ 0 & \delta & \beta\delta \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \left(-1+e^{\delta}\right)\alpha & -\alpha\beta+e^{\delta}\alpha\beta+\gamma-\alpha\beta\delta \\ 0 & e^{\delta} & \left(-1+e^{\delta}\right)\beta \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

```
MatrixExp[α1 x + β1 y + γ1 z + δ1 t].MatrixExp[α2 x + β2 y + γ2 z + δ2 t]  // MatrixForm
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$$\begin{pmatrix} 1 & \frac{e^{\delta_2}\left(-1+e^{\delta_1}\right)\alpha_1}{\delta_1}+\frac{\left(-1+e^{\delta_2}\right)\alpha_2}{\delta_2} & \frac{-\alpha_1\beta_1+e^{\delta_1}\alpha_1\beta_1-\alpha_1\beta_1\delta_1+\gamma_1\delta_1^2}{\delta_1^2}+\frac{\left(-1+e^{\delta_1}\right)\left(-1+e^{\delta_2}\right)\alpha_1\beta_2}{\delta_1\delta_2}+\frac{-\alpha_2\beta_2+e^{\delta_2}\alpha_2\beta_2-\alpha_2\beta_2\delta_2+\gamma_2\delta_2^2}{\delta_2^2} \\ 0 & e^{\delta_1+\delta_2} & \frac{\left(-1+e^{\delta_1}\right)\beta_1}{\delta_1}+\frac{e^{\delta_1}\left(-1+e^{\delta_2}\right)\beta_2}{\delta_2} \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixExp[α1 x + β1 y + γ1 z + δ1 t].MatrixExp[α2 x + β2 y + γ2 z + δ2 t]  // MatrixLog //
   PowerExpand // FullSimplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{(\delta_1+\delta_2)\left(\left(-1+e^{\delta_2}\right)\alpha_2\delta_1+e^{\delta_2}\left(-1+e^{\delta_1}\right)\alpha_1\delta_2\right)}{\left(-1+e^{\delta_1+\delta_2}\right)\delta_1\delta_2} & -\left(\left(-1+e^{\delta_2}\right)\alpha_2\delta_1+e^{\delta_2}\left(-1+e^{\delta_1}\right)\alpha_1\delta_2\right)\left(e^{\delta_1}\left(-1+e^{\delta_2}\right)\beta_2\delta_1+\left(-1+e^{\delta_1}\right)\beta_1\delta_2\right)+\frac{(\delta_1+\delta_2)\left(\left(-1+e^{\delta_2}\right)\alpha_2 \cdots\right.}{\cdots} \cdots \\ 0 & \delta_1+\delta_2 & \\ 0 & 0 & \end{pmatrix}$$

```
MatrixExp[δ1 α1 x + δ1 β1 y + γ1 z + δ1 t].MatrixExp[δ2 α2 x + δ2 β2 y + γ2 z + δ2 t]  // MatrixLog //
   PowerExpand // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\left(e^{\delta_2}\left(-1+e^{\delta_1}\right)\alpha_1+\left(-1+e^{\delta_2}\right)\alpha_2\right)(\delta_1+\delta_2)}{-1+e^{\delta_1+\delta_2}} & \frac{-\alpha_1\beta_1+e^{\delta_1}\alpha_1\beta_1+e^{\delta_2}\alpha_1\beta_1-e^{\delta_1+\delta_2}\alpha_1\beta_1+\alpha_2\beta_1-e^{\delta_1}\alpha_2\beta_1-e^{\delta_2}\alpha_2\beta_1+e^{\delta_1+\delta_2}\alpha_2\beta_1+\alpha_1\beta_2-e^{\delta_1}\alpha_1\beta_2-e^{\delta_2}\alpha_1 \cdots}{\cdots} \\ 0 & \delta_1+\delta_2 & \\ 0 & 0 & \end{pmatrix}$$

```
MatrixExp[α1 x + β1 y + γ1 z].MatrixExp[α2 x + β2 y + γ2 z]  // MatrixLog // PowerExpand //
  FullSimplify // MatrixForm
```

$$\begin{pmatrix} 0 & \alpha_1+\alpha_2 & -\frac{1}{2}\alpha_2\beta_1+\frac{\alpha_1\beta_2}{2}+\gamma_1+\gamma_2 \\ 0 & 0 & \beta_1+\beta_2 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixExp[α1 x + β1 y].MatrixExp[α2 x + β2 y]  // MatrixLog // PowerExpand // FullSimplify //
 MatrixForm
```

$$\begin{pmatrix} 0 & \alpha_1+\alpha_2 & \frac{1}{2}(-\alpha_2\beta_1+\alpha_1\beta_2) \\ 0 & 0 & \beta_1+\beta_2 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixExp[α1 x + δ1 t].MatrixExp[α2 x + δ2 t]  // MatrixLog // PowerExpand // FullSimplify //
 MatrixForm
```

$$\begin{pmatrix} 0 & \frac{(\delta_1+\delta_2)\left(\left(-1+e^{\delta_2}\right)\alpha_2\delta_1+e^{\delta_2}\left(-1+e^{\delta_1}\right)\alpha_1\delta_2\right)}{\left(-1+e^{\delta_1+\delta_2}\right)\delta_1\delta_2} & 0 \\ 0 & \delta_1+\delta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixExp[δ1 α1 x + δ1 t].MatrixExp[δ2 α2 x + δ2 t]  // MatrixLog // PowerExpand //
  FullSimplify // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\left(-\alpha_2+e^{\delta_2}\left(\left(-1+e^{\delta_1}\right)\alpha_1+\alpha_2\right)\right)\left(\delta_1+\delta_2\right)}{-1+e^{\delta_1+\delta_2}} & 0 \\ 0 & \delta_1+\delta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$