Cheat Sheet Quantum Groups

Every Formula Worth Dreading

[Ma1, Example 1.3.2]. A Hopf algebra over $\mathbb{Q}[q^{\pm 1}]$:

$$\langle 1, X, g^{\pm 1}\rangle \Big/ \left(Xg^{\pm 1} = q^{\pm 1} g^{\pm 1} X\right),$$

with $\Delta\colon (X,\ g^{\pm 1}) \mapsto (X\otimes 1 + g\otimes X,\ g^{\pm 1}\otimes g^{\pm 1})$, $\epsilon\colon (X,\ g^{\pm 1}) \mapsto (0,\ 1)$, and $S : (X,\ g^{\pm 1}) \mapsto (-g^{-1}X,\ g^{\mp 1})$. Has $S^2 \neq 1$.

[Ma2, Example 18.8] (seems to have no universal R formula):

Example 18.8 *Given a Cartan and root datum and $H, B, \tilde{B}$ as above, and $q \in k^*$ such that $q_i^2 \neq 1$ for all i, we have an ordinary Hopf algebra $U_q = B' \bowtie H \bowtie B^{\rm op}$. It consists of generators e_i, f_i, K_μ, the relations*

$$K_\mu K_\nu = K_{\mu+\nu}, \quad K_\mu e^i = q^{\langle \mu, i'\rangle} e^i K_\mu, \quad f_i K_\mu = q^{\langle \mu, i'\rangle} K_\mu f_i,$$
$$[e^i, f_j] = \frac{K_i^{\frac{i\cdot i}{2}} - K_i^{-\frac{i\cdot i}{2}}}{q_i - q_i^{-1}} \delta^i_j$$

and the relations coming from the kernels of ev. *The coalgebra is*

$$\Delta e^i = e^i \otimes K_i^{\frac{i\cdot i}{2}} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + K_i^{-\frac{i\cdot i}{2}} \otimes f_i,$$
$$\epsilon(K_\mu) = 1, \quad \epsilon(e^i) = \epsilon(f_i) = 0.$$

The antipode has the usual form uniquely determined by the above.

[CP, Theorem 8.3.9]:

THEOREM 8.3.9 *For any finite-dimensional complex simple Lie algebra $\mathfrak{g}$, $U_h(\mathfrak{g})$ is topologically quasitriangular with universal R-matrix*

$$\mathcal{R}_h = \exp\left[h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j\right] \prod_\beta \exp_{q_\beta}[(1 - q_\beta^{-2}) X_\beta^+ \otimes X_\beta^-],$$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that the β_r-term appears to the left of the β_s-term if $r > s$ (see 8.1.4). ∎

References.

[CP] V. Chari and A. Pressley, *A Guide to Quantum Groups,* Cambridge University Press, 1994.

[Ka] C. Kassel, *Quantum groups,* Springer-Verlag GTM **155**, Heidelberg 1994.

[Ma1] S. Majid, *Foundations of Quantum Group Theory,* Cambridge University Press, 1995.

[Ma2] S. Majid, *A Quantum Groups Primer,* London Mathematical Society Lecture Note Series **292**, Cambridge University Press, 2002.