

Pensieve header: Implementing and verifying  $\$gl\_n^\wedge\epsilon\$,$  version 2.

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 $\chi_{cond}$  := If[TrueQ@cond, 1, 0];
B[0, _] = 0; B[_ , 0] = 0;
B[c_*x : (e | f | g | h)_, y_] := Expand[c B[x, y]];
B[y_, c_*x : (e | f | g | h)_] := Expand[c B[y, x]];
B[x_Plus, y_] := B[# , y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
P[0, _] = 0; P[_ , 0] = 0;
P[c_*x : (e | f | g | h)_, y_] := Expand[c P[x, y]];
P[y_, c_*x : (e | f | g | h)_] := Expand[c P[y, x]];
P[x_Plus, y_] := P[# , y] & /@ x;
P[x_, y_Plus] := P[x, #] & /@ y;

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P[ei,j_, fk,l_] :=  $\chi_{j=k \wedge i=l}$ ; P[fk,l_, ei,j_] :=  $\chi_{j=k \wedge i=l}$ ;
P[e_ , h_] = 0; P[h_ , e_] = 0;
P[g_ , f_] = 0; P[f_ , g_] = 0;
P[gi_, hj_] :=  $\chi_{i=j}$ ; P[hj_, gi_] :=  $\chi_{i=j}$ ;
P[(e | g)_, (e | g)_] = 0;
P[(f | h)_, (f | h)_] = 0;

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B[(h | g)_, (h | g)_] = 0;

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B[ei,j_, ek,l_] :=  $\chi_{j=k} e_{i,l} - \chi_{l=i} e_{k,j}$ ;
B[gi_, ej,k_] :=  $(\chi_{i=j} - \chi_{i=k}) e_{j,k}$ ;

B[fi,j_, fk,l_] :=  $\epsilon \chi_{j=k} f_{i,l} - \epsilon \chi_{l=i} f_{k,j}$ ;
B[hi_, fj,k_] :=  $\epsilon / 2 (\chi_{i=j} - \chi_{i=k}) f_{j,k}$ ;

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B[ei,j_, hk_] :=  $-\epsilon / 2 (\chi_{i=k} - \chi_{j=k}) e_{i,j}$ ;
B[gi_, fj,k_] :=  $(\chi_{i=j} - \chi_{i=k}) f_{j,k}$ ;
B[ei,j_, fk,l_] := Expand[
 $\chi_{j=k} (\epsilon \chi_{i<l} e_{i,l} + \chi_{i=l} (h_i + \epsilon / 2 g_i) + \chi_{i>l} f_{i,l}) -$ 
 $\chi_{l=i} (\epsilon \chi_{k<j} e_{k,j} + \chi_{k=j} (h_j + \epsilon / 2 g_j) + \chi_{k>j} f_{k,j})$ ];

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B[ej,k_, gi_] := Expand[-B[gi_, ej,k]];
B[fj,k_, hi_] := Expand[-B[hi_, fj,k]];
B[hk_, ei,j_] := Expand[-B[ei,j_, hk]];
B[fj,k_, gi_] := Expand[-B[gi_, fj,k]];
B[fi,j_, ek,l_] := Expand[-B[ek,l_, fi,j]];

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Basis+[n_] := Union@Flatten@{
  Table[ei,j, {i, n - 1}, {j, i + 1, n}],
  Table[gi, {i, n}]
};
Basis-[n_] := Union@Flatten@{
  Table[fi,j, {i, 2, n}, {j, i - 1}],
  Table[hi, {i, n}]
};
Basis[n_] := Join[Basis+[n], Basis-[n]];

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**Basis[4]**

{g<sub>1</sub>, g<sub>2</sub>, g<sub>3</sub>, g<sub>4</sub>, e<sub>1,2</sub>, e<sub>1,3</sub>, e<sub>1,4</sub>, e<sub>2,3</sub>, e<sub>2,4</sub>, e<sub>3,4</sub>, h<sub>1</sub>, h<sub>2</sub>, h<sub>3</sub>, h<sub>4</sub>, f<sub>2,1</sub>, f<sub>3,1</sub>, f<sub>3,2</sub>, f<sub>4,1</sub>, f<sub>4,2</sub>, f<sub>4,3</sub>}

n = 3;

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Table[
  {x, y} → B[x, y],
  {x, Basis[n]}, {y, Basis[n]}
] // MatrixForm

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{g <sub>1</sub> , g <sub>1</sub> } → 0	{g <sub>1</sub> , g <sub>2</sub> } → 0	{g <sub>1</sub> , g <sub>3</sub> } → 0	{g <sub>1</sub> , e <sub>1,2</sub> } → e <sub>1,2</sub>	
{g <sub>2</sub> , g <sub>1</sub> } → 0	{g <sub>2</sub> , g <sub>2</sub> } → 0	{g <sub>2</sub> , g <sub>3</sub> } → 0	{g <sub>2</sub> , e <sub>1,2</sub> } → -e <sub>1,2</sub>	
{g <sub>3</sub> , g <sub>1</sub> } → 0	{g <sub>3</sub> , g <sub>2</sub> } → 0	{g <sub>3</sub> , g <sub>3</sub> } → 0	{g <sub>3</sub> , e <sub>1,2</sub> } → 0	{
{e <sub>1,2</sub> , g <sub>1</sub> } → -e <sub>1,2</sub>	{e <sub>1,2</sub> , g <sub>2</sub> } → e <sub>1,2</sub>	{e <sub>1,2</sub> , g <sub>3</sub> } → 0	{e <sub>1,2</sub> , e <sub>1,2</sub> } → 0	
{e <sub>1,3</sub> , g <sub>1</sub> } → -e <sub>1,3</sub>	{e <sub>1,3</sub> , g <sub>2</sub> } → 0	{e <sub>1,3</sub> , g <sub>3</sub> } → e <sub>1,3</sub>	{e <sub>1,3</sub> , e <sub>1,2</sub> } → 0	
{e <sub>2,3</sub> , g <sub>1</sub> } → 0	{e <sub>2,3</sub> , g <sub>2</sub> } → -e <sub>2,3</sub>	{e <sub>2,3</sub> , g <sub>3</sub> } → e <sub>2,3</sub>	{e <sub>2,3</sub> , e <sub>1,2</sub> } → -e <sub>1,3</sub>	
{h <sub>1</sub> , g <sub>1</sub> } → 0	{h <sub>1</sub> , g <sub>2</sub> } → 0	{h <sub>1</sub> , g <sub>3</sub> } → 0	{h <sub>1</sub> , e <sub>1,2</sub> } → $\frac{1}{2} \in e_{1,2}$	{h
{h <sub>2</sub> , g <sub>1</sub> } → 0	{h <sub>2</sub> , g <sub>2</sub> } → 0	{h <sub>2</sub> , g <sub>3</sub> } → 0	{h <sub>2</sub> , e <sub>1,2</sub> } → $-\frac{1}{2} \in e_{1,2}$	
{h <sub>3</sub> , g <sub>1</sub> } → 0	{h <sub>3</sub> , g <sub>2</sub> } → 0	{h <sub>3</sub> , g <sub>3</sub> } → 0	{h <sub>3</sub> , e <sub>1,2</sub> } → 0	{h <sub>3</sub>
{f <sub>2,1</sub> , g <sub>1</sub> } → f <sub>2,1</sub>	{f <sub>2,1</sub> , g <sub>2</sub> } → -f <sub>2,1</sub>	{f <sub>2,1</sub> , g <sub>3</sub> } → 0	{f <sub>2,1</sub> , e <sub>1,2</sub> } → $-\frac{e_{g_1}}{2} + \frac{e_{g_2}}{2} - h_1 + h_2$	{f
{f <sub>3,1</sub> , g <sub>1</sub> } → f <sub>3,1</sub>	{f <sub>3,1</sub> , g <sub>2</sub> } → 0	{f <sub>3,1</sub> , g <sub>3</sub> } → -f <sub>3,1</sub>	{f <sub>3,1</sub> , e <sub>1,2</sub> } → f <sub>3,2</sub>	{f <sub>3,1</sub> , e <sub>1</sub>
{f <sub>3,2</sub> , g <sub>1</sub> } → 0	{f <sub>3,2</sub> , g <sub>2</sub> } → f <sub>3,2</sub>	{f <sub>3,2</sub> , g <sub>3</sub> } → -f <sub>3,2</sub>	{f <sub>3,2</sub> , e <sub>1,2</sub> } → 0	{f <sub>3</sub>



**DS[2, 1, 3]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \epsilon h_1 - \epsilon h_2, \right. \\ \left. \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, -\frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\epsilon h_1 + \epsilon h_2, -2 \in e_{1,2}, 2 \in e_{1,2}, -2 \in f_{2,1}, 2 \in f_{2,1} \right\}$$

**DS[2, 1, 4]**

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**DS[2, 2, 1]**

$$\{ g_1, g_2, e_{1,2}, h_1, h_2, f_{2,1}, \epsilon g_1, \epsilon g_2, \epsilon e_{1,2}, \\ \epsilon h_1, \epsilon h_2, \epsilon f_{2,1}, \epsilon^2 g_1, \epsilon^2 g_2, \epsilon^2 e_{1,2}, \epsilon^2 h_1, \epsilon^2 h_2, \epsilon^2 f_{2,1} \}$$

**DS[2, 2, 2]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \right. \\ -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \epsilon^2 h_1 - \epsilon^2 h_2, -\epsilon^2 h_1 + \epsilon^2 h_2, -e_{1,2}, e_{1,2}, -\epsilon e_{1,2}, \\ -\frac{1}{2} \epsilon e_{1,2}, \frac{1}{2} \epsilon e_{1,2}, \epsilon e_{1,2}, -\epsilon^2 e_{1,2}, -\frac{1}{2} \epsilon^2 e_{1,2}, \frac{1}{2} \epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, -f_{2,1}, f_{2,1}, \\ \left. -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \epsilon f_{2,1}, -\epsilon^2 f_{2,1}, -\frac{1}{2} \epsilon^2 f_{2,1}, \frac{1}{2} \epsilon^2 f_{2,1}, \epsilon^2 f_{2,1} \right\}$$

**DS[2, 2, 3]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \frac{\epsilon^2 g_1}{4} - \frac{\epsilon^2 g_2}{4} + \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, \right. \\ -\frac{1}{4} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{4} - \frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \epsilon^2 h_1 - \epsilon^2 h_2, \frac{\epsilon^2 h_1}{2} - \frac{\epsilon^2 h_2}{2}, \\ \frac{\epsilon^2 h_1}{4} - \frac{\epsilon^2 h_2}{4}, -\frac{1}{4} \epsilon^2 h_1 + \frac{\epsilon^2 h_2}{4}, -\frac{1}{2} \epsilon^2 h_1 + \frac{\epsilon^2 h_2}{2}, -\epsilon^2 h_1 + \epsilon^2 h_2, -2 \in e_{1,2}, 2 \in e_{1,2}, -2 \epsilon^2 e_{1,2}, \\ \left. -\epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, 2 \epsilon^2 e_{1,2}, -2 \in f_{2,1}, 2 \in f_{2,1}, -2 \epsilon^2 f_{2,1}, -\epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, 2 \epsilon^2 f_{2,1} \right\}$$

**DS[2, 2, 4]**

$$\{ 4 \epsilon^2 h_1 - 4 \epsilon^2 h_2, -4 \epsilon^2 h_1 + 4 \epsilon^2 h_2, -4 \epsilon^2 e_{1,2}, 4 \epsilon^2 e_{1,2}, -4 \epsilon^2 f_{2,1}, 4 \epsilon^2 f_{2,1} \}$$

**DS[2, 2, 5]**

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**DS[2, 3, 1]**

$$\{ g_1, g_2, e_{1,2}, h_1, h_2, f_{2,1}, \epsilon g_1, \epsilon g_2, \epsilon e_{1,2}, \epsilon h_1, \epsilon h_2, \epsilon f_{2,1}, \epsilon^2 g_1, \\ \epsilon^2 g_2, \epsilon^2 e_{1,2}, \epsilon^2 h_1, \epsilon^2 h_2, \epsilon^2 f_{2,1}, \epsilon^3 g_1, \epsilon^3 g_2, \epsilon^3 e_{1,2}, \epsilon^3 h_1, \epsilon^3 h_2, \epsilon^3 f_{2,1} \}$$

**DS[2, 3, 2]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \right. \\ \left. \frac{\epsilon^3 g_1}{2} - \frac{\epsilon^3 g_2}{2} + \epsilon^2 h_1 - \epsilon^2 h_2, -\frac{1}{2} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{2} - \epsilon^2 h_1 + \epsilon^2 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, -\epsilon^3 h_1 + \epsilon^3 h_2, \right. \\ \left. -e_{1,2}, e_{1,2}, -\epsilon e_{1,2}, -\frac{1}{2} \epsilon e_{1,2}, \frac{1}{2} \epsilon e_{1,2}, \epsilon e_{1,2}, -\epsilon^2 e_{1,2}, -\frac{1}{2} \epsilon^2 e_{1,2}, \frac{1}{2} \epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, \right. \\ \left. -\epsilon^3 e_{1,2}, -\frac{1}{2} \epsilon^3 e_{1,2}, \frac{1}{2} \epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, -f_{2,1}, f_{2,1}, -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \right. \\ \left. \epsilon f_{2,1}, -\epsilon^2 f_{2,1}, -\frac{1}{2} \epsilon^2 f_{2,1}, \frac{1}{2} \epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, -\epsilon^3 f_{2,1}, -\frac{1}{2} \epsilon^3 f_{2,1}, \frac{1}{2} \epsilon^3 f_{2,1}, \epsilon^3 f_{2,1} \right\}$$

**DS[2, 3, 3]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \frac{\epsilon^2 g_1}{4} - \frac{\epsilon^2 g_2}{4} + \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, \right. \\ \left. -\frac{1}{4} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{4} - \frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \frac{\epsilon^3 g_1}{2} - \frac{\epsilon^3 g_2}{2} + \epsilon^2 h_1 - \epsilon^2 h_2, \right. \\ \left. \frac{\epsilon^3 g_1}{4} - \frac{\epsilon^3 g_2}{4} + \frac{\epsilon^2 h_1}{2} - \frac{\epsilon^2 h_2}{2}, \frac{\epsilon^3 g_1}{8} - \frac{\epsilon^3 g_2}{8} + \frac{\epsilon^2 h_1}{4} - \frac{\epsilon^2 h_2}{4}, -\frac{1}{8} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{8} - \frac{\epsilon^2 h_1}{4} + \frac{\epsilon^2 h_2}{4}, \right. \\ \left. -\frac{1}{4} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{4} - \frac{\epsilon^2 h_1}{2} + \frac{\epsilon^2 h_2}{2}, -\frac{1}{2} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{2} - \epsilon^2 h_1 + \epsilon^2 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, \frac{\epsilon^3 h_1}{2} - \frac{\epsilon^3 h_2}{2}, \right. \\ \left. \frac{\epsilon^3 h_1}{4} - \frac{\epsilon^3 h_2}{4}, -\frac{1}{4} \epsilon^3 h_1 + \frac{\epsilon^3 h_2}{4}, -\frac{1}{2} \epsilon^3 h_1 + \frac{\epsilon^3 h_2}{2}, -\epsilon^3 h_1 + \epsilon^3 h_2, -2 \epsilon e_{1,2}, 2 \epsilon e_{1,2}, \right. \\ \left. -2 \epsilon^2 e_{1,2}, -\epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, 2 \epsilon^2 e_{1,2}, -2 \epsilon^3 e_{1,2}, -\epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, 2 \epsilon^3 e_{1,2}, -2 \epsilon f_{2,1}, \right. \\ \left. 2 \epsilon f_{2,1}, -2 \epsilon^2 f_{2,1}, -\epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, 2 \epsilon^2 f_{2,1}, -2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{2,1}, \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1} \right\}$$

**DS[2, 3, 4]**

$$\left\{ 2 \epsilon^3 g_1 - 2 \epsilon^3 g_2 + 4 \epsilon^2 h_1 - 4 \epsilon^2 h_2, -2 \epsilon^3 g_1 + 2 \epsilon^3 g_2 - 4 \epsilon^2 h_1 + 4 \epsilon^2 h_2, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_2, \right. \\ \left. 2 \epsilon^3 h_1 - 2 \epsilon^3 h_2, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_2, -4 \epsilon^3 h_1 + 4 \epsilon^3 h_2, -4 \epsilon^2 e_{1,2}, 4 \epsilon^2 e_{1,2}, -4 \epsilon^3 e_{1,2}, \right. \\ \left. -2 \epsilon^3 e_{1,2}, 2 \epsilon^3 e_{1,2}, 4 \epsilon^3 e_{1,2}, -4 \epsilon^2 f_{2,1}, 4 \epsilon^2 f_{2,1}, -4 \epsilon^3 f_{2,1}, -2 \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1}, 4 \epsilon^3 f_{2,1} \right\}$$

**DS[2, 3, 5]**

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**DS[2, 10, 7]**

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**DS[3, 1, 1]**

$$\{ g_1, g_2, g_3, e_{1,2}, e_{1,3}, e_{2,3}, h_1, h_2, h_3, f_{2,1}, f_{3,1}, f_{3,2}, \epsilon g_1, \\ \epsilon g_2, \epsilon g_3, \epsilon e_{1,2}, \epsilon e_{1,3}, \epsilon e_{2,3}, \epsilon h_1, \epsilon h_2, \epsilon h_3, \epsilon f_{2,1}, \epsilon f_{3,1}, \epsilon f_{3,2} \}$$

**DS[3, 1, 2]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \epsilon h_1 - \epsilon h_2, -\epsilon h_1 + \epsilon h_2, \frac{\epsilon g_1}{2} - \frac{\epsilon g_3}{2} + h_1 - h_3, \right. \\ \left. \frac{\epsilon g_2}{2} - \frac{\epsilon g_3}{2} + h_2 - h_3, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_3}{2} - h_1 + h_3, -\frac{\epsilon g_2}{2} + \frac{\epsilon g_3}{2} - h_2 + h_3, \epsilon h_1 - \epsilon h_3, \right. \\ \left. \epsilon h_2 - \epsilon h_3, -\epsilon h_1 + \epsilon h_3, -\epsilon h_2 + \epsilon h_3, -e_{1,2}, e_{1,2}, -\epsilon e_{1,2}, -\frac{1}{2} \epsilon e_{1,2}, \frac{1}{2} \epsilon e_{1,2}, \epsilon e_{1,2}, \right. \\ \left. -e_{1,3}, e_{1,3}, -\epsilon e_{1,3}, -\frac{1}{2} \epsilon e_{1,3}, \frac{1}{2} \epsilon e_{1,3}, \epsilon e_{1,3}, -e_{2,3}, e_{2,3}, -\epsilon e_{2,3}, -\frac{1}{2} \epsilon e_{2,3}, \right. \\ \left. \frac{1}{2} \epsilon e_{2,3}, \epsilon e_{2,3}, -f_{2,1}, f_{2,1}, -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \epsilon f_{2,1}, -f_{3,1}, f_{3,1}, \right. \\ \left. -\epsilon f_{3,1}, -\frac{1}{2} \epsilon f_{3,1}, \frac{1}{2} \epsilon f_{3,1}, \epsilon f_{3,1}, -f_{3,2}, f_{3,2}, -\epsilon f_{3,2}, -\frac{1}{2} \epsilon f_{3,2}, \frac{1}{2} \epsilon f_{3,2}, \epsilon f_{3,2} \right\}$$

**DS[3, 1, 3]**

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \epsilon h_1 - \epsilon h_2, \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, -\frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, \right. \\ \left. -\epsilon h_1 + \epsilon h_2, \frac{\epsilon g_1}{2} - \frac{\epsilon g_3}{2} + h_1 - h_3, \frac{\epsilon g_2}{2} - \frac{\epsilon g_3}{2} + h_2 - h_3, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_3}{2} - h_1 + h_3, \right. \\ \left. -\frac{\epsilon g_2}{2} + \frac{\epsilon g_3}{2} - h_2 + h_3, \epsilon h_1 - \epsilon h_3, \epsilon h_2 - \epsilon h_3, \frac{\epsilon h_1}{2} - \frac{\epsilon h_3}{2}, \frac{\epsilon h_2}{2} - \frac{\epsilon h_3}{2}, -\frac{\epsilon h_1}{2} + \frac{\epsilon h_3}{2}, \right. \\ \left. -\frac{\epsilon h_2}{2} + \frac{\epsilon h_3}{2}, -\epsilon h_1 + \epsilon h_3, -\epsilon h_2 + \epsilon h_3, -2 \epsilon e_{1,2}, -\epsilon e_{1,2}, \epsilon e_{1,2}, 2 \epsilon e_{1,2}, -e_{1,3}, e_{1,3}, \right. \\ \left. -2 \epsilon e_{1,3}, -\epsilon e_{1,3}, -\frac{1}{2} \epsilon e_{1,3}, \frac{1}{2} \epsilon e_{1,3}, \epsilon e_{1,3}, 2 \epsilon e_{1,3}, -2 \epsilon e_{2,3}, -\epsilon e_{2,3}, \epsilon e_{2,3}, 2 \epsilon e_{2,3}, \right. \\ \left. -f_{2,1}, f_{2,1}, -2 \epsilon f_{2,1}, -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \epsilon f_{2,1}, 2 \epsilon f_{2,1}, -2 \epsilon f_{3,1}, -\epsilon f_{3,1}, \right. \\ \left. \epsilon f_{3,1}, 2 \epsilon f_{3,1}, -f_{3,2}, f_{3,2}, -2 \epsilon f_{3,2}, -\epsilon f_{3,2}, -\frac{1}{2} \epsilon f_{3,2}, \frac{1}{2} \epsilon f_{3,2}, \epsilon f_{3,2}, 2 \epsilon f_{3,2} \right\}$$

**DS[3, 1, 4]**

$$\left\{ 2 \epsilon h_1 - 2 \epsilon h_2, \epsilon h_1 - \epsilon h_2, -\epsilon h_1 + \epsilon h_2, -2 \epsilon h_1 + 2 \epsilon h_2, 2 \epsilon h_1 - 2 \epsilon h_3, \right. \\ \left. 2 \epsilon h_2 - 2 \epsilon h_3, \epsilon h_1 - \epsilon h_3, \epsilon h_2 - \epsilon h_3, -\epsilon h_1 + \epsilon h_3, -\epsilon h_2 + \epsilon h_3, -2 \epsilon h_1 + 2 \epsilon h_3, \right. \\ \left. -2 \epsilon h_2 + 2 \epsilon h_3, -\epsilon e_{1,2}, \epsilon e_{1,2}, -2 \epsilon e_{1,3}, -\epsilon e_{1,3}, \epsilon e_{1,3}, 2 \epsilon e_{1,3}, -\epsilon e_{2,3}, \epsilon e_{2,3}, \right. \\ \left. -2 \epsilon f_{2,1}, -\epsilon f_{2,1}, \epsilon f_{2,1}, 2 \epsilon f_{2,1}, -\epsilon f_{3,1}, \epsilon f_{3,1}, -2 \epsilon f_{3,2}, -\epsilon f_{3,2}, \epsilon f_{3,2}, 2 \epsilon f_{3,2} \right\}$$

**DS[3, 1, 5]**

{}

**DS[5, 1, 5]**

$$\left\{ \epsilon h_1 - \epsilon h_2, -\epsilon h_1 + \epsilon h_2, \epsilon h_2 - \epsilon h_3, -\epsilon h_2 + \epsilon h_3, \epsilon h_3 - \epsilon h_4, -\epsilon h_3 + \epsilon h_4, \epsilon h_1 - \epsilon h_5, \epsilon h_4 - \epsilon h_5, \right. \\ \left. -\epsilon h_1 + \epsilon h_5, -\epsilon h_4 + \epsilon h_5, -\epsilon e_{1,4}, \epsilon e_{1,4}, -\epsilon e_{1,5}, \epsilon e_{1,5}, -\epsilon e_{2,5}, \epsilon e_{2,5}, -\epsilon f_{2,1}, \epsilon f_{2,1}, \right. \\ \left. -\epsilon f_{3,1}, \epsilon f_{3,1}, -\epsilon f_{3,2}, \epsilon f_{3,2}, -\epsilon f_{4,2}, \epsilon f_{4,2}, -\epsilon f_{4,3}, \epsilon f_{4,3}, -\epsilon f_{5,3}, \epsilon f_{5,3}, -\epsilon f_{5,4}, \epsilon f_{5,4} \right\}$$

**DS[5, 1, 6]**

{}

**DS[5, 2, 6]**

{}

**DS[5, 3, 6]**

$$\{2 \epsilon^3 h_1 - 2 \epsilon^3 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, -\epsilon^3 h_1 + \epsilon^3 h_2, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_2, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_3, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_3, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_3, \epsilon^3 h_1 - \epsilon^3 h_3, \epsilon^3 h_2 - \epsilon^3 h_3, -\epsilon^3 h_1 + \epsilon^3 h_3, -\epsilon^3 h_2 + \epsilon^3 h_3, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_3, \\ -2 \epsilon^3 h_2 + 2 \epsilon^3 h_3, -4 \epsilon^3 h_1 + 4 \epsilon^3 h_3, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_4, 4 \epsilon^3 h_2 - 4 \epsilon^3 h_4, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_4, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_4, 2 \epsilon^3 h_3 - 2 \epsilon^3 h_4, \epsilon^3 h_1 - \epsilon^3 h_4, \epsilon^3 h_2 - \epsilon^3 h_4, \epsilon^3 h_3 - \epsilon^3 h_4, -\epsilon^3 h_1 + \epsilon^3 h_4, \\ -\epsilon^3 h_2 + \epsilon^3 h_4, -\epsilon^3 h_3 + \epsilon^3 h_4, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_4, -2 \epsilon^3 h_2 + 2 \epsilon^3 h_4, -2 \epsilon^3 h_3 + 2 \epsilon^3 h_4, \\ -4 \epsilon^3 h_1 + 4 \epsilon^3 h_4, -4 \epsilon^3 h_2 + 4 \epsilon^3 h_4, 4 \epsilon^3 h_2 - 4 \epsilon^3 h_5, 4 \epsilon^3 h_3 - 4 \epsilon^3 h_5, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_5, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_5, 2 \epsilon^3 h_3 - 2 \epsilon^3 h_5, 2 \epsilon^3 h_4 - 2 \epsilon^3 h_5, \epsilon^3 h_1 - \epsilon^3 h_5, \epsilon^3 h_2 - \epsilon^3 h_5, \epsilon^3 h_3 - \epsilon^3 h_5, \\ \epsilon^3 h_4 - \epsilon^3 h_5, -\epsilon^3 h_1 + \epsilon^3 h_5, -\epsilon^3 h_2 + \epsilon^3 h_5, -\epsilon^3 h_3 + \epsilon^3 h_5, -\epsilon^3 h_4 + \epsilon^3 h_5, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_5, \\ -2 \epsilon^3 h_2 + 2 \epsilon^3 h_5, -2 \epsilon^3 h_3 + 2 \epsilon^3 h_5, -2 \epsilon^3 h_4 + 2 \epsilon^3 h_5, -4 \epsilon^3 h_2 + 4 \epsilon^3 h_5, -4 \epsilon^3 h_3 + 4 \epsilon^3 h_5, \\ -\epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, -\epsilon^3 e_{1,3}, \epsilon^3 e_{1,3}, -\epsilon^3 e_{1,4}, \epsilon^3 e_{1,4}, -2 \epsilon^3 e_{1,5}, -\epsilon^3 e_{1,5}, \epsilon^3 e_{1,5}, 2 \epsilon^3 e_{1,5}, \\ -\epsilon^3 e_{2,3}, \epsilon^3 e_{2,3}, -\epsilon^3 e_{2,4}, \epsilon^3 e_{2,4}, -\epsilon^3 e_{2,5}, \epsilon^3 e_{2,5}, -\epsilon^3 e_{3,4}, \epsilon^3 e_{3,4}, -\epsilon^3 e_{3,5}, \epsilon^3 e_{3,5}, \\ -\epsilon^3 e_{4,5}, \epsilon^3 e_{4,5}, -2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{2,1}, \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{3,1}, \epsilon^3 f_{3,1}, -2 \epsilon^3 f_{3,2}, -\epsilon^3 f_{3,2}, \\ \epsilon^3 f_{3,2}, 2 \epsilon^3 f_{3,2}, -\epsilon^3 f_{4,1}, \epsilon^3 f_{4,1}, -\epsilon^3 f_{4,2}, \epsilon^3 f_{4,2}, -2 \epsilon^3 f_{4,3}, -\epsilon^3 f_{4,3}, \epsilon^3 f_{4,3}, 2 \epsilon^3 f_{4,3}, \\ -\epsilon^3 f_{5,1}, \epsilon^3 f_{5,1}, -\epsilon^3 f_{5,2}, \epsilon^3 f_{5,2}, -\epsilon^3 f_{5,3}, \epsilon^3 f_{5,3}, -2 \epsilon^3 f_{5,4}, -\epsilon^3 f_{5,4}, \epsilon^3 f_{5,4}, 2 \epsilon^3 f_{5,4}\}$$

**DS[5, 3, 7]**

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