

Pensieve header: Implementing and verifying $\$gl_n^\epsilon\$$.

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 $\chi$ [cond_] := If[TrueQ@cond, 1, 0];
(*B /: (B[x_,y_]=z_) := (B[x,y]:=z; B[y,x]:=-z);*)
B[0, _] = 0; B[_ , 0] = 0;
B[c_*x : (e | f | g | h)_, y_] := Expand[c B[x, y]];
B[y_, c_*x : (e | f | g | h)_] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
P[0, _] = 0; P[_ , 0] = 0;
P[c_*x : (e | f | g | h)_, y_] := Expand[c P[x, y]];
P[y_, c_*x : (e | f | g | h)_] := Expand[c P[y, x]];
P[x_Plus, y_] := P[#, y] & /@ x;
P[x_, y_Plus] := P[x, #] & /@ y;

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P[e_{i,j}, f_{k,l}] :=  $\chi$ [j == k ^ i == l]; P[f_{k,l}, e_{i,j}] :=  $\chi$ [j == k ^ i == l];
P[e_, h_] = 0; P[h_, e_] = 0;
P[g_, f_] = 0; P[f_, g_] = 0;
P[g_i, h_j] :=  $\chi$ [i == j]; P[h_j, g_i] :=  $\chi$ [i == j];
P[(e | g)_, (e | g)_] = 0;
P[(f | h)_, (f | h)_] = 0;

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{s1 -> -1/2, s2 -> -1/2, s3 -> 1, s4 -> 1, s5 -> -1/2,
 s6 -> -1/2, s7 -> 1, s8 -> 1, s9 -> 1, s10 -> 1, s11 -> 1} /. Rule -> Set;
B[e_{i,j}, e_{k,l}] :=  $\chi$ [j == k] e_{i,l} -  $\chi$ [l == i] e_{k,j};
B[g_i, e_{j,k}] := ( $\chi$ [i == j] -  $\chi$ [i == k]) e_{j,k}; B[e_{j,k}, g_i] := -( $\chi$ [i == j] -  $\chi$ [i == k]) e_{j,k};
B[g_, g_] = 0;

B[f_{i,j}, f_{k,l}] :=  $\epsilon$   $\chi$ [j == k] f_{i,l} -  $\epsilon$   $\chi$ [l == i] f_{k,j};
B[h_i, f_{j,k}] := - $\epsilon$  s1 ( $\chi$ [i == j] -  $\chi$ [i == k]) f_{j,k};
B[f_{j,k}, h_i] :=  $\epsilon$  s1 ( $\chi$ [i == j] -  $\chi$ [i == k]) f_{j,k};
B[h_, h_] = 0;

B[g_, h_] = 0; B[h_, g_] = 0;
B[e_{i,j}, h_k] := s2  $\epsilon$  ( $\chi$ [i == k] -  $\chi$ [j == k]) e_{i,j};
B[h_k, e_{i,j}] := - $\epsilon$  s2 ( $\chi$ [i == k] -  $\chi$ [j == k]) e_{i,j};
B[g_i, f_{j,k}] := s11 ( $\chi$ [i == j] -  $\chi$ [i == k]) f_{j,k};
B[f_{j,k}, g_i] := -s11 ( $\chi$ [i == j] -  $\chi$ [i == k]) f_{j,k};
B[e_{i,j}, f_{k,l}] :=
  Expand[s3  $\chi$ [j == k] ( $\chi$ [i < l]  $\epsilon$  e_{i,l} +  $\chi$ [i == l] (s7 h_i - s5  $\epsilon$  g_i) + s9  $\chi$ [i > l] f_{i,l}) -
    s4 s3  $\chi$ [l == i] ( $\chi$ [k < j]  $\epsilon$  e_{k,j} +  $\chi$ [k == j] (s8 h_j - s6  $\epsilon$  g_j) + s10  $\chi$ [k > j] f_{k,j})];
B[f_{i,j}, e_{k,l}] := Expand[-B[e_{k,l}, f_{i,j}]]

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Basis+[n_] := Union@Flatten@{
  Table[ei,j, {i, n - 1}, {j, i + 1, n}],
  Table[gi, {i, n}]
};
Basis-[n_] := Union@Flatten@{
  Table[fi,j, {i, 2, n}, {j, i - 1}],
  Table[hi, {i, n}]
};
Basis[n_] := Join[Basis+[n], Basis-[n]];

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Basis[4]

{g₁, g₂, g₃, g₄, e_{1,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{2,4}, e_{3,4}, h₁, h₂, h₃, h₄, f_{2,1}, f_{3,1}, f_{3,2}, f_{4,1}, f_{4,2}, f_{4,3}}

n = 3;

Table[

{x, y} → B[x, y],
 {x, Basis[n]}, {y, Basis[n]}
] // MatrixForm

{g ₁ , g ₁ } → 0	{g ₁ , g ₂ } → 0	{g ₁ , g ₃ } → 0	{g ₁ , e _{1,2} } → e _{1,2}	
{g ₂ , g ₁ } → 0	{g ₂ , g ₂ } → 0	{g ₂ , g ₃ } → 0	{g ₂ , e _{1,2} } → -e _{1,2}	
{g ₃ , g ₁ } → 0	{g ₃ , g ₂ } → 0	{g ₃ , g ₃ } → 0	{g ₃ , e _{1,2} } → 0	{
{e _{1,2} , g ₁ } → -e _{1,2}	{e _{1,2} , g ₂ } → e _{1,2}	{e _{1,2} , g ₃ } → 0	{e _{1,2} , e _{1,2} } → 0	
{e _{1,3} , g ₁ } → -e _{1,3}	{e _{1,3} , g ₂ } → 0	{e _{1,3} , g ₃ } → e _{1,3}	{e _{1,3} , e _{1,2} } → 0	
{e _{2,3} , g ₁ } → 0	{e _{2,3} , g ₂ } → -e _{2,3}	{e _{2,3} , g ₃ } → e _{2,3}	{e _{2,3} , e _{1,2} } → -e _{1,3}	
{h ₁ , g ₁ } → 0	{h ₁ , g ₂ } → 0	{h ₁ , g ₃ } → 0	{h ₁ , e _{1,2} } → $\frac{1}{2} \in e_{1,2}$	{h
{h ₂ , g ₁ } → 0	{h ₂ , g ₂ } → 0	{h ₂ , g ₃ } → 0	{h ₂ , e _{1,2} } → $-\frac{1}{2} \in e_{1,2}$	
{h ₃ , g ₁ } → 0	{h ₃ , g ₂ } → 0	{h ₃ , g ₃ } → 0	{h ₃ , e _{1,2} } → 0	{h ₃
{f _{2,1} , g ₁ } → f _{2,1}	{f _{2,1} , g ₂ } → -f _{2,1}	{f _{2,1} , g ₃ } → 0	{f _{2,1} , e _{1,2} } → $-\frac{e_{g_1}}{2} + \frac{e_{g_2}}{2} - h_1 + h_2$	{f
{f _{3,1} , g ₁ } → f _{3,1}	{f _{3,1} , g ₂ } → 0	{f _{3,1} , g ₃ } → -f _{3,1}	{f _{3,1} , e _{1,2} } → f _{3,2}	{f _{3,1} , e ₁
{f _{3,2} , g ₁ } → 0	{f _{3,2} , g ₂ } → f _{3,2}	{f _{3,2} , g ₃ } → -f _{3,2}	{f _{3,2} , e _{1,2} } → 0	{f ₃

DS[2, 1, 3]

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \epsilon h_1 - \epsilon h_2, \right. \\ \left. \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, -\frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\epsilon h_1 + \epsilon h_2, -2 \in e_{1,2}, 2 \in e_{1,2}, -2 \in f_{2,1}, 2 \in f_{2,1} \right\}$$

DS[2, 1, 4]

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DS[2, 2, 1]

$$\{ g_1, g_2, e_{1,2}, h_1, h_2, f_{2,1}, \epsilon g_1, \epsilon g_2, \epsilon e_{1,2}, \\ \epsilon h_1, \epsilon h_2, \epsilon f_{2,1}, \epsilon^2 g_1, \epsilon^2 g_2, \epsilon^2 e_{1,2}, \epsilon^2 h_1, \epsilon^2 h_2, \epsilon^2 f_{2,1} \}$$

DS[2, 2, 2]

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \right. \\ -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \epsilon^2 h_1 - \epsilon^2 h_2, -\epsilon^2 h_1 + \epsilon^2 h_2, -e_{1,2}, e_{1,2}, -\epsilon e_{1,2}, \\ -\frac{1}{2} \epsilon e_{1,2}, \frac{1}{2} \epsilon e_{1,2}, \epsilon e_{1,2}, -\epsilon^2 e_{1,2}, -\frac{1}{2} \epsilon^2 e_{1,2}, \frac{1}{2} \epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, -f_{2,1}, f_{2,1}, \\ \left. -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \epsilon f_{2,1}, -\epsilon^2 f_{2,1}, -\frac{1}{2} \epsilon^2 f_{2,1}, \frac{1}{2} \epsilon^2 f_{2,1}, \epsilon^2 f_{2,1} \right\}$$

DS[2, 2, 3]

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \frac{\epsilon^2 g_1}{4} - \frac{\epsilon^2 g_2}{4} + \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, \right. \\ -\frac{1}{4} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{4} - \frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \epsilon^2 h_1 - \epsilon^2 h_2, \frac{\epsilon^2 h_1}{2} - \frac{\epsilon^2 h_2}{2}, \\ \frac{\epsilon^2 h_1}{4} - \frac{\epsilon^2 h_2}{4}, -\frac{1}{4} \epsilon^2 h_1 + \frac{\epsilon^2 h_2}{4}, -\frac{1}{2} \epsilon^2 h_1 + \frac{\epsilon^2 h_2}{2}, -\epsilon^2 h_1 + \epsilon^2 h_2, -2 \in e_{1,2}, 2 \in e_{1,2}, -2 \epsilon^2 e_{1,2}, \\ \left. -\epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, 2 \epsilon^2 e_{1,2}, -2 \in f_{2,1}, 2 \in f_{2,1}, -2 \epsilon^2 f_{2,1}, -\epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, 2 \epsilon^2 f_{2,1} \right\}$$

DS[2, 2, 4]

$$\{ 4 \epsilon^2 h_1 - 4 \epsilon^2 h_2, -4 \epsilon^2 h_1 + 4 \epsilon^2 h_2, -4 \epsilon^2 e_{1,2}, 4 \epsilon^2 e_{1,2}, -4 \epsilon^2 f_{2,1}, 4 \epsilon^2 f_{2,1} \}$$

DS[2, 2, 5]

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DS[2, 3, 1]

$$\{ g_1, g_2, e_{1,2}, h_1, h_2, f_{2,1}, \epsilon g_1, \epsilon g_2, \epsilon e_{1,2}, \epsilon h_1, \epsilon h_2, \epsilon f_{2,1}, \epsilon^2 g_1, \\ \epsilon^2 g_2, \epsilon^2 e_{1,2}, \epsilon^2 h_1, \epsilon^2 h_2, \epsilon^2 f_{2,1}, \epsilon^3 g_1, \epsilon^3 g_2, \epsilon^3 e_{1,2}, \epsilon^3 h_1, \epsilon^3 h_2, \epsilon^3 f_{2,1} \}$$

DS[2, 3, 2]

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \right. \\ \left. \frac{\epsilon^3 g_1}{2} - \frac{\epsilon^3 g_2}{2} + \epsilon^2 h_1 - \epsilon^2 h_2, -\frac{1}{2} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{2} - \epsilon^2 h_1 + \epsilon^2 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, -\epsilon^3 h_1 + \epsilon^3 h_2, \right. \\ \left. -e_{1,2}, e_{1,2}, -\epsilon e_{1,2}, -\frac{1}{2} \epsilon e_{1,2}, \frac{1}{2} \epsilon e_{1,2}, \epsilon e_{1,2}, -\epsilon^2 e_{1,2}, -\frac{1}{2} \epsilon^2 e_{1,2}, \frac{1}{2} \epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, \right. \\ \left. -\epsilon^3 e_{1,2}, -\frac{1}{2} \epsilon^3 e_{1,2}, \frac{1}{2} \epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, -f_{2,1}, f_{2,1}, -\epsilon f_{2,1}, -\frac{1}{2} \epsilon f_{2,1}, \frac{1}{2} \epsilon f_{2,1}, \right. \\ \left. \epsilon f_{2,1}, -\epsilon^2 f_{2,1}, -\frac{1}{2} \epsilon^2 f_{2,1}, \frac{1}{2} \epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, -\epsilon^3 f_{2,1}, -\frac{1}{2} \epsilon^3 f_{2,1}, \frac{1}{2} \epsilon^3 f_{2,1}, \epsilon^3 f_{2,1} \right\}$$

DS[2, 3, 3]

$$\left\{ \frac{\epsilon g_1}{2} - \frac{\epsilon g_2}{2} + h_1 - h_2, -\frac{\epsilon g_1}{2} + \frac{\epsilon g_2}{2} - h_1 + h_2, \frac{\epsilon^2 g_1}{2} - \frac{\epsilon^2 g_2}{2} + \epsilon h_1 - \epsilon h_2, \frac{\epsilon^2 g_1}{4} - \frac{\epsilon^2 g_2}{4} + \frac{\epsilon h_1}{2} - \frac{\epsilon h_2}{2}, \right. \\ \left. -\frac{1}{4} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{4} - \frac{\epsilon h_1}{2} + \frac{\epsilon h_2}{2}, -\frac{1}{2} \epsilon^2 g_1 + \frac{\epsilon^2 g_2}{2} - \epsilon h_1 + \epsilon h_2, \frac{\epsilon^3 g_1}{2} - \frac{\epsilon^3 g_2}{2} + \epsilon^2 h_1 - \epsilon^2 h_2, \right. \\ \left. \frac{\epsilon^3 g_1}{4} - \frac{\epsilon^3 g_2}{4} + \frac{\epsilon^2 h_1}{2} - \frac{\epsilon^2 h_2}{2}, \frac{\epsilon^3 g_1}{8} - \frac{\epsilon^3 g_2}{8} + \frac{\epsilon^2 h_1}{4} - \frac{\epsilon^2 h_2}{4}, -\frac{1}{8} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{8} - \frac{\epsilon^2 h_1}{4} + \frac{\epsilon^2 h_2}{4}, \right. \\ \left. -\frac{1}{4} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{4} - \frac{\epsilon^2 h_1}{2} + \frac{\epsilon^2 h_2}{2}, -\frac{1}{2} \epsilon^3 g_1 + \frac{\epsilon^3 g_2}{2} - \epsilon^2 h_1 + \epsilon^2 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, \frac{\epsilon^3 h_1}{2} - \frac{\epsilon^3 h_2}{2}, \right. \\ \left. \frac{\epsilon^3 h_1}{4} - \frac{\epsilon^3 h_2}{4}, -\frac{1}{4} \epsilon^3 h_1 + \frac{\epsilon^3 h_2}{4}, -\frac{1}{2} \epsilon^3 h_1 + \frac{\epsilon^3 h_2}{2}, -\epsilon^3 h_1 + \epsilon^3 h_2, -2 \epsilon e_{1,2}, 2 \epsilon e_{1,2}, \right. \\ \left. -2 \epsilon^2 e_{1,2}, -\epsilon^2 e_{1,2}, \epsilon^2 e_{1,2}, 2 \epsilon^2 e_{1,2}, -2 \epsilon^3 e_{1,2}, -\epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, 2 \epsilon^3 e_{1,2}, -2 \epsilon f_{2,1}, \right. \\ \left. 2 \epsilon f_{2,1}, -2 \epsilon^2 f_{2,1}, -\epsilon^2 f_{2,1}, \epsilon^2 f_{2,1}, 2 \epsilon^2 f_{2,1}, -2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{2,1}, \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1} \right\}$$

DS[2, 3, 4]

$$\left\{ 2 \epsilon^3 g_1 - 2 \epsilon^3 g_2 + 4 \epsilon^2 h_1 - 4 \epsilon^2 h_2, -2 \epsilon^3 g_1 + 2 \epsilon^3 g_2 - 4 \epsilon^2 h_1 + 4 \epsilon^2 h_2, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_2, \right. \\ \left. 2 \epsilon^3 h_1 - 2 \epsilon^3 h_2, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_2, -4 \epsilon^3 h_1 + 4 \epsilon^3 h_2, -4 \epsilon^2 e_{1,2}, 4 \epsilon^2 e_{1,2}, -4 \epsilon^3 e_{1,2}, \right. \\ \left. -2 \epsilon^3 e_{1,2}, 2 \epsilon^3 e_{1,2}, 4 \epsilon^3 e_{1,2}, -4 \epsilon^2 f_{2,1}, 4 \epsilon^2 f_{2,1}, -4 \epsilon^3 f_{2,1}, -2 \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1}, 4 \epsilon^3 f_{2,1} \right\}$$

DS[2, 3, 5]

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DS[2, 10, 7]

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DS[3, 1, 1]

$$\left\{ g_1, g_2, g_3, e_{1,2}, e_{1,3}, e_{2,3}, h_1, h_2, h_3, f_{2,1}, f_{3,1}, f_{3,2}, \epsilon g_1, \right. \\ \left. \epsilon g_2, \epsilon g_3, \epsilon e_{1,2}, \epsilon e_{1,3}, \epsilon e_{2,3}, \epsilon h_1, \epsilon h_2, \epsilon h_3, \epsilon f_{2,1}, \epsilon f_{3,1}, \epsilon f_{3,2} \right\}$$

DS[3, 1, 2]

$$\left\{ \begin{aligned} & \frac{\in g_1}{2} - \frac{\in g_2}{2} + h_1 - h_2, -\frac{\in g_1}{2} + \frac{\in g_2}{2} - h_1 + h_2, \in h_1 - \in h_2, -\in h_1 + \in h_2, \frac{\in g_1}{2} - \frac{\in g_3}{2} + h_1 - h_3, \\ & \frac{\in g_2}{2} - \frac{\in g_3}{2} + h_2 - h_3, -\frac{\in g_1}{2} + \frac{\in g_3}{2} - h_1 + h_3, -\frac{\in g_2}{2} + \frac{\in g_3}{2} - h_2 + h_3, \in h_1 - \in h_3, \\ & \in h_2 - \in h_3, -\in h_1 + \in h_3, -\in h_2 + \in h_3, -e_{1,2}, e_{1,2}, -\in e_{1,2}, -\frac{1}{2} \in e_{1,2}, \frac{1}{2} \in e_{1,2}, \in e_{1,2}, \\ & -e_{1,3}, e_{1,3}, -\in e_{1,3}, -\frac{1}{2} \in e_{1,3}, \frac{1}{2} \in e_{1,3}, \in e_{1,3}, -e_{2,3}, e_{2,3}, -\in e_{2,3}, -\frac{1}{2} \in e_{2,3}, \\ & \frac{1}{2} \in e_{2,3}, \in e_{2,3}, -f_{2,1}, f_{2,1}, -\in f_{2,1}, -\frac{1}{2} \in f_{2,1}, \frac{1}{2} \in f_{2,1}, \in f_{2,1}, -f_{3,1}, f_{3,1}, \\ & -\in f_{3,1}, -\frac{1}{2} \in f_{3,1}, \frac{1}{2} \in f_{3,1}, \in f_{3,1}, -f_{3,2}, f_{3,2}, -\in f_{3,2}, -\frac{1}{2} \in f_{3,2}, \frac{1}{2} \in f_{3,2}, \in f_{3,2} \end{aligned} \right\}$$

DS[3, 1, 3]

$$\left\{ \begin{aligned} & \frac{\in g_1}{2} - \frac{\in g_2}{2} + h_1 - h_2, -\frac{\in g_1}{2} + \frac{\in g_2}{2} - h_1 + h_2, \in h_1 - \in h_2, \frac{\in h_1}{2} - \frac{\in h_2}{2}, -\frac{\in h_1}{2} + \frac{\in h_2}{2}, \\ & -\in h_1 + \in h_2, \frac{\in g_1}{2} - \frac{\in g_3}{2} + h_1 - h_3, \frac{\in g_2}{2} - \frac{\in g_3}{2} + h_2 - h_3, -\frac{\in g_1}{2} + \frac{\in g_3}{2} - h_1 + h_3, \\ & -\frac{\in g_2}{2} + \frac{\in g_3}{2} - h_2 + h_3, \in h_1 - \in h_3, \in h_2 - \in h_3, \frac{\in h_1}{2} - \frac{\in h_3}{2}, \frac{\in h_2}{2} - \frac{\in h_3}{2}, -\frac{\in h_1}{2} + \frac{\in h_3}{2}, \\ & -\frac{\in h_2}{2} + \frac{\in h_3}{2}, -\in h_1 + \in h_3, -\in h_2 + \in h_3, -2 \in e_{1,2}, -\in e_{1,2}, \in e_{1,2}, 2 \in e_{1,2}, -e_{1,3}, e_{1,3}, \\ & -2 \in e_{1,3}, -\in e_{1,3}, -\frac{1}{2} \in e_{1,3}, \frac{1}{2} \in e_{1,3}, \in e_{1,3}, 2 \in e_{1,3}, -2 \in e_{2,3}, -\in e_{2,3}, \in e_{2,3}, 2 \in e_{2,3}, \\ & -f_{2,1}, f_{2,1}, -2 \in f_{2,1}, -\in f_{2,1}, -\frac{1}{2} \in f_{2,1}, \frac{1}{2} \in f_{2,1}, \in f_{2,1}, 2 \in f_{2,1}, -2 \in f_{3,1}, -\in f_{3,1}, \\ & \in f_{3,1}, 2 \in f_{3,1}, -f_{3,2}, f_{3,2}, -2 \in f_{3,2}, -\in f_{3,2}, -\frac{1}{2} \in f_{3,2}, \frac{1}{2} \in f_{3,2}, \in f_{3,2}, 2 \in f_{3,2} \end{aligned} \right\}$$

DS[3, 1, 4]

$$\left\{ \begin{aligned} & 2 \in h_1 - 2 \in h_2, \in h_1 - \in h_2, -\in h_1 + \in h_2, -2 \in h_1 + 2 \in h_2, 2 \in h_1 - 2 \in h_3, \\ & 2 \in h_2 - 2 \in h_3, \in h_1 - \in h_3, \in h_2 - \in h_3, -\in h_1 + \in h_3, -\in h_2 + \in h_3, -2 \in h_1 + 2 \in h_3, \\ & -2 \in h_2 + 2 \in h_3, -\in e_{1,2}, \in e_{1,2}, -2 \in e_{1,3}, -\in e_{1,3}, \in e_{1,3}, 2 \in e_{1,3}, -\in e_{2,3}, \in e_{2,3}, \\ & -2 \in f_{2,1}, -\in f_{2,1}, \in f_{2,1}, 2 \in f_{2,1}, -\in f_{3,1}, \in f_{3,1}, -2 \in f_{3,2}, -\in f_{3,2}, \in f_{3,2}, 2 \in f_{3,2} \end{aligned} \right\}$$

DS[3, 1, 5]

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DS[5, 1, 5]

$$\left\{ \begin{aligned} & \in h_1 - \in h_2, -\in h_1 + \in h_2, \in h_2 - \in h_3, -\in h_2 + \in h_3, \in h_3 - \in h_4, -\in h_3 + \in h_4, \in h_1 - \in h_5, \in h_4 - \in h_5, \\ & -\in h_1 + \in h_5, -\in h_4 + \in h_5, -\in e_{1,4}, \in e_{1,4}, -\in e_{1,5}, \in e_{1,5}, -\in e_{2,5}, \in e_{2,5}, -\in f_{2,1}, \in f_{2,1}, \\ & -\in f_{3,1}, \in f_{3,1}, -\in f_{3,2}, \in f_{3,2}, -\in f_{4,2}, \in f_{4,2}, -\in f_{4,3}, \in f_{4,3}, -\in f_{5,3}, \in f_{5,3}, -\in f_{5,4}, \in f_{5,4} \end{aligned} \right\}$$

DS[5, 1, 6]

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DS[5, 2, 6]

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DS[5, 3, 6]

$$\{2 \epsilon^3 h_1 - 2 \epsilon^3 h_2, \epsilon^3 h_1 - \epsilon^3 h_2, -\epsilon^3 h_1 + \epsilon^3 h_2, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_2, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_3, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_3, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_3, \epsilon^3 h_1 - \epsilon^3 h_3, \epsilon^3 h_2 - \epsilon^3 h_3, -\epsilon^3 h_1 + \epsilon^3 h_3, -\epsilon^3 h_2 + \epsilon^3 h_3, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_3, \\ -2 \epsilon^3 h_2 + 2 \epsilon^3 h_3, -4 \epsilon^3 h_1 + 4 \epsilon^3 h_3, 4 \epsilon^3 h_1 - 4 \epsilon^3 h_4, 4 \epsilon^3 h_2 - 4 \epsilon^3 h_4, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_4, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_4, 2 \epsilon^3 h_3 - 2 \epsilon^3 h_4, \epsilon^3 h_1 - \epsilon^3 h_4, \epsilon^3 h_2 - \epsilon^3 h_4, \epsilon^3 h_3 - \epsilon^3 h_4, -\epsilon^3 h_1 + \epsilon^3 h_4, \\ -\epsilon^3 h_2 + \epsilon^3 h_4, -\epsilon^3 h_3 + \epsilon^3 h_4, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_4, -2 \epsilon^3 h_2 + 2 \epsilon^3 h_4, -2 \epsilon^3 h_3 + 2 \epsilon^3 h_4, \\ -4 \epsilon^3 h_1 + 4 \epsilon^3 h_4, -4 \epsilon^3 h_2 + 4 \epsilon^3 h_4, 4 \epsilon^3 h_2 - 4 \epsilon^3 h_5, 4 \epsilon^3 h_3 - 4 \epsilon^3 h_5, 2 \epsilon^3 h_1 - 2 \epsilon^3 h_5, \\ 2 \epsilon^3 h_2 - 2 \epsilon^3 h_5, 2 \epsilon^3 h_3 - 2 \epsilon^3 h_5, 2 \epsilon^3 h_4 - 2 \epsilon^3 h_5, \epsilon^3 h_1 - \epsilon^3 h_5, \epsilon^3 h_2 - \epsilon^3 h_5, \epsilon^3 h_3 - \epsilon^3 h_5, \\ \epsilon^3 h_4 - \epsilon^3 h_5, -\epsilon^3 h_1 + \epsilon^3 h_5, -\epsilon^3 h_2 + \epsilon^3 h_5, -\epsilon^3 h_3 + \epsilon^3 h_5, -\epsilon^3 h_4 + \epsilon^3 h_5, -2 \epsilon^3 h_1 + 2 \epsilon^3 h_5, \\ -2 \epsilon^3 h_2 + 2 \epsilon^3 h_5, -2 \epsilon^3 h_3 + 2 \epsilon^3 h_5, -2 \epsilon^3 h_4 + 2 \epsilon^3 h_5, -4 \epsilon^3 h_2 + 4 \epsilon^3 h_5, -4 \epsilon^3 h_3 + 4 \epsilon^3 h_5, \\ -\epsilon^3 e_{1,2}, \epsilon^3 e_{1,2}, -\epsilon^3 e_{1,3}, \epsilon^3 e_{1,3}, -\epsilon^3 e_{1,4}, \epsilon^3 e_{1,4}, -2 \epsilon^3 e_{1,5}, -\epsilon^3 e_{1,5}, \epsilon^3 e_{1,5}, 2 \epsilon^3 e_{1,5}, \\ -\epsilon^3 e_{2,3}, \epsilon^3 e_{2,3}, -\epsilon^3 e_{2,4}, \epsilon^3 e_{2,4}, -\epsilon^3 e_{2,5}, \epsilon^3 e_{2,5}, -\epsilon^3 e_{3,4}, \epsilon^3 e_{3,4}, -\epsilon^3 e_{3,5}, \epsilon^3 e_{3,5}, \\ -\epsilon^3 e_{4,5}, \epsilon^3 e_{4,5}, -2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{2,1}, \epsilon^3 f_{2,1}, 2 \epsilon^3 f_{2,1}, -\epsilon^3 f_{3,1}, \epsilon^3 f_{3,1}, -2 \epsilon^3 f_{3,2}, -\epsilon^3 f_{3,2}, \\ \epsilon^3 f_{3,2}, 2 \epsilon^3 f_{3,2}, -\epsilon^3 f_{4,1}, \epsilon^3 f_{4,1}, -\epsilon^3 f_{4,2}, \epsilon^3 f_{4,2}, -2 \epsilon^3 f_{4,3}, -\epsilon^3 f_{4,3}, \epsilon^3 f_{4,3}, 2 \epsilon^3 f_{4,3}, \\ -\epsilon^3 f_{5,1}, \epsilon^3 f_{5,1}, -\epsilon^3 f_{5,2}, \epsilon^3 f_{5,2}, -\epsilon^3 f_{5,3}, \epsilon^3 f_{5,3}, -2 \epsilon^3 f_{5,4}, -\epsilon^3 f_{5,4}, \epsilon^3 f_{5,4}, 2 \epsilon^3 f_{5,4}\}$$

DS[5, 3, 7]

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